Quantitative recurrence properties of expanding maps

J.L. Fernández*
Universidad Autónoma de Madrid
Madrid , Spain
joseluis.fernandez@uam.es

M.V. Melián*
Universidad Autónoma de Madrid
Madrid , Spain
mavi.melian@uam.es

D. Pestana †
Universidad Carlos III de Madrid
Madrid, Spain
domingo.pestana@uc3m.es

Abstract

Under a map T, a point x recurs at rate given by a sequence $\{r_n\}$ near a point x_0 if $d(T^n(x), x_0) < r_n$ infinitely often. Let us fix x_0, and consider the set of those x's. In this paper, we study the size of this set for expanding maps and obtain its measure and sharp lower bounds on its dimension involving the entropy of T, the local dimension near x_0 and the upper limit of $\frac{1}{n} \log \frac{1}{r_n}$. We apply our results in several concrete examples including subshifts of finite type, Gauss transformation and inner functions.

1 Introduction

The pre-images under a mixing transformation T distribute themselves somehow regularly along the base space. In this paper we aim to quantify this regularity by studying both the measure and dimension of some recurrence sets. More precisely, we study the behaviour of pre-images under expanding transformations, i.e. transformations which locally increase distances.

Throughout this paper (X, d) will be a locally complete separable metric space endowed with a finite measure λ over the σ-algebra $\mathcal{A}$ of Borel sets. We further assume throughout that the support of λ is equal to X and that λ is a non-atomic measure. We recall that a measurable transformation $T : X \longrightarrow X$ preserves the measure λ if $\lambda(T^{-1}(A)) = \lambda(A)$ for every $A \in \mathcal{A}$. The classical recurrence theorem of Poincaré (see, for example, [23], p.61) says that

Theorem A (H. Poincaré). *If $T : X \longrightarrow X$ preserves the measure λ, then λ-almost every point of X is recurrent, in the sense that*

$$\liminf_{n\to\infty} d(T^n(x), x) = 0 \,.$$

Here and hereafter T^n denotes the n-th fold composition $T^n = T \circ T \circ \cdots \circ T$. M. Boshernitzan obtained in [8] the following quantitative version of Theorem A.

Theorem B (M. Boshernitzan). *If the Hausdorff α-measure H_α on X is σ-finite for some $\alpha > 0$ and $T : X \longrightarrow X$ preserves the measure λ, then for λ-almost all $x \in X$,*

$$\liminf_{n\to\infty} n^{1/\alpha}\, d(T^n(x), x) < \infty \,.$$

[0]Mathematics Subject Classification (2000): 37D05, 37A05, 37A25, 37F10, 28D05,11K55, 11K60, 30D05, 30D50.

Keywords: Quantitative recurrence, expanding maps, Hausdorff dimension, diophantine approximation, Gauss transformation, inner functions.

*Research supported by Grant BFM2003-04780 from Ministerio de Ciencia y Tecnología, Spain

†Research supported by Grants BFM2003-04780 and BFM2003-06335-C03-02 Ministerio de Ciencia y Tecnología, Spain

Besides, if $H_\alpha(X) = 0$, then for λ-almost all $x \in X$,

$$\liminf_{n\to\infty} n^{1/\alpha}\, d(T^n(x), x) = 0 \tag{1}$$

and when the measure λ agrees with H_α for some $\alpha > 0$, then for λ-almost all $x \in X$,

$$\liminf_{n\to\infty} n^{1/\alpha}\, d(T^n(x), x) \le 1\,.$$

L. Barreira and B. Saussol [3] have obtained a generalization of (1) when $X \subseteq \mathbf{R}^N$ in terms of the lower pointwise dimension of λ at the point $x \in X$ instead of the Hausdorff measure of X and other authors have also obtained new quantitative recurrence results relating various recurrence indicators with entropy and dimension, see e.g. [2], [6], [24], [25] and [41].

It is natural to ask if the orbit $\{T^n(x)\}$ of the point x comes back not only to every neighborhood of x itself as Poincaré's Theorem asserts, but whether it also visits every neighborhood of a previously chosen point $x_0 \in X$. Under the additional hypothesis of ergodicity it is easy to check that for any $x_0 \in X$, we have that

$$\liminf_{n\to\infty} d(T^n(x), x_0) = 0\,, \qquad \text{for } \lambda\text{-almost all } x \in X. \tag{2}$$

Recall that the transformation T is *ergodic* if the only T-invariant sets (up to sets of λ-measure zero) are trivial, *i.e.* they have zero λ-measure or their complements have zero λ-measure.

In order to obtain a quantitative version of (2) along the lines of Theorem B we need stronger mixing properties on T. In [19] we studied *uniformly mixing* transformations. For these transformations we obtained, for example, that, given a decreasing sequence $\{r_n\}$ of positive numbers tending to zero as $n \to \infty$, if

$$\sum_{n=1}^{\infty} \lambda(B(x_0, r_n)) = \infty,$$

then

$$\lim_{n\to\infty} \frac{\#\{i \le n : \ d(T^i(x), x_0) \le r_i\,\}}{\sum_{j=1}^{n} \lambda(B(x_0, r_j))} = 1\,, \qquad \text{for } \lambda\text{-almost every } x \in X\,,$$

and therefore

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \le 1\,, \qquad \text{for } \lambda\text{-almost every } x \in X\,.$$

Here and hereafter the notation $\#A$ means the number of elements of the set A.

Expanding maps.

In this paper we consider the recurrence properties of the orbits under expanding maps, a context which encompasses many interesting examples: subshift of finite type, in particular Bernoulli shifts, Gauss transformation and continued fractions expansions, some inner functions and expanding endomorphisms of compact manifolds.

An expanding map $T : X \longrightarrow X$ does not, in general, preserves the given measure λ for a given expanding system, but among all the measures invariant under T there exists a unique probability measure μ which is locally absolutely continuous with respect to λ and has good mixing properties (see Theorem E). We will refer to μ as the absolutely continuous invariant probability measure (ACIPM). For the complete definition of expanding maps we refer to Section 4. However we will describe here their main properties in an informal way. An expanding system (X, d, λ, T) has an associated Markov partition $\mathcal{P}_0$ of X in such a way that T is injective in each block P of $\mathcal{P}_0$ and $T(P)$ is a union of blocks of $\mathcal{P}_0$. Also there exists a positive measurable function $\mathbf{J}$ on X, the *Jacobian* of T, such that

$$\lambda(T(A)) = \int_A \mathbf{J}\, d\lambda\,, \qquad \text{if } A \text{ is contained in some block of } \mathcal{P}_0$$

and $\mathbf{J}$ has the following distortion property for all x, y in the same block of $\mathcal{P}_0$:

$$\left| \frac{\mathbf{J}(x)}{\mathbf{J}(y)} - 1 \right| \le C_1\, d(T(x), T(y))^\alpha\,.$$

Here $C_1 > 0$ and $0 < \alpha \le 1$ are absolute constants. This property allows us to compare the ratio $\lambda(A)/\lambda(A')$ with $\lambda(T(A))/\lambda(T(A'))$for A, A' contained in the same block of $\mathcal{P}_0$.

Finally, the reason for the name expanding is the property that if x, y belong to the same block of the partition $\mathcal{P}_n = \vee_{j=0}^n T^{-j}(\mathcal{P}_0)$, then

$$d(T^n(x), T^n(y)) \ge C_2 \beta^n d(x, y)$$

with absolute constants $C_2 > 0$ and $\beta > 1$.

In this paper we are interested in studying for expanding systems the size of the set of points of X where $\liminf_{n\to\infty} d(T^n(x), x_0)/r_n = 0$, where $\{r_n\}$ is a given sequence of positive numbers and x_0 is a previously chosen point in X. To do this we study the size of the set

$$\mathcal{W}(x_0, \{r_n\}) = \{x \in X : \ d(T^n(x), x_0) < r_n \text{ for infinitely many } n\}\,.$$

Our first objective is to study the relationship between the measure of this set and how fast goes to zero the sequence of radii. We use the following definitions of local dimension.

Definition 1.1. *The lower and upper $\mathcal{P}_0$-dimension of the measure μ at the point $x \in X$ are defined, respectively, by*

$$\underline{\delta}_\mu(x) = \liminf_{n\to\infty} \frac{\log \mu(P(n,x))}{\log \operatorname{diam}(P(n,x))}\,, \qquad \overline{\delta}_\mu(x) = \limsup_{n\to\infty} \frac{\log \mu(P(n,x))}{\log \operatorname{diam}(P(n,x))}\,.$$

Here and hereafter $P(n,x)$ denotes the block of the partition $\mathcal{P}_n = \vee_{j=0}^n T^{-j}(\mathcal{P}_0)$ which contains the point $x \in X$.

We have the following result.

Theorem 1.1. *Let (X, d, λ, T) be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$. Let $\{r_n\}$ be a non increasing sequence of positive numbers. Then for λ-almost all point $x_0 \in X$ we have that*

$$\text{if} \qquad \sum_{n=1}^{\infty} r_n^\delta = \infty \quad \text{for some } \delta > \overline{\delta}_\lambda(x_0), \quad \text{then} \quad \mathcal{W}(x_0, \{r_n\}) \text{ has full } \lambda\text{-measure}\,,$$

and we can conclude that, for λ-almost all point $x_0 \in X$,

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0\,, \qquad \text{for } \lambda\text{-almost all } x \in X\,.$$

In particular, we have, for $\alpha > \overline{\delta}_\lambda(x_0)$, that for λ-almost all point $x_0 \in X$

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n(x), x_0) = 0\,, \qquad \text{for } \lambda\text{-almost all } x \in X\,.$$

Further results about the points x_0 which satisfy the conclusions in Theorem 1.1 are included in Section 5. There is also a quantitative version when the system has the Bernoulli property (see Theorem 5.2).

If the sequence $\{r_n\}$ tends to zero in such a way that $\sum_n \lambda(B(x_0, r_n)) < \infty$ then it is easy to check that $\lambda(\mathcal{W}(x_0, \{r_n\})) = 0$ and therefore

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \ge 1\,, \qquad \text{for } \lambda\text{-almost all } x \in X\,.$$

As a consequence we get that if $\lambda(B(x_0, r)) \le C\, r^\Delta$ for all r, then for $\alpha < \Delta$

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n(x), x_0) = \infty\,, \qquad \text{for } \lambda\text{-almost all } x \in X\,.$$

Theorem 1.1 is sharp in the sense that if the series $\sum_{n=1}^\infty r_n^\delta$ diverges for $\delta < \overline{\delta}_\lambda(x_0)$, then it can happen that the set $\mathcal{W}(x_0, \{r_n\})$ has zero λ-measure. Consider, for instance, the sequence $r_n = 1/n^{1/\delta}$ with $\delta < 1$, and an expanding system with $X \subset \mathbf{R}$ and λ the Lebesgue measure. Then $\overline{\delta}_\lambda(x_0) = 1$ and $\sum_n r_n^\delta = \infty$. But $\sum_n \lambda(B(x_0, r_n)) < \infty$ and therefore $\lambda(\mathcal{W}(x_0, \{r_n\})) = 0$. On the other hand, for expanding systems with extra mixing properties, Theorem 1.1 still holds for $\delta = \overline{\delta}_\lambda(x_0)$ (see, Theorem 3 in [19]).

Even in the case that the set $\mathcal{W}(x_0, \{r_n\})$ has zero λ-measure we have proved that this set is large since we have obtained a positive lower bound for its dimension.

In this paper we use two different notions of dimension: the λ-grid dimension ($\mathrm{Dim}_{\mathbf{\Pi},\lambda}$), considering coverings with blocks of the partitions $\mathcal{P}_n$, and the λ-Hausdorff dimension (Dim_λ), when we consider coverings with balls of small diameter (see Section 2 for the definitions). We remark that Dim_λ is equal to $1/N$ times the usual Hausdorff dimension when λ is the Lebesgue measure in $X = \mathbf{R}^N$. To obtain lower bounds for the dimension, we have constructed a Cantor-like set contained in $\mathcal{W}(x_0, \{r_n\})$. The elements of the different families of the Cantor set are certain blocks of some partitions $\mathcal{P}_n$. In our construction these blocks have controlled μ-measure and they are in a certain sense well distributed. The main difficulty while estimating the dimension of this Cantor set is that does not have a fixed pattern and the ratio between the measure of a 'parent' and his 'son' can be very big depending on the sequence of radii. Our approach is contained in Theorem 2.1.

The main tools in the construction of the Cantor set are: (1) good estimates for the measure μ of some blocks of $\mathcal{P}_n$, obtained as a consequence of Shannon-McMillan-Breimann Theorem (see Theorem D); (2) good estimates of the ratio between $\lambda(P(n+1,x))$ and $\lambda(P(n,x))$ due to the distortion property of $\mathbf{J}$. An extra difficulty is that the measures λ and μ are only comparable in each block of the partition $\mathcal{P}_0$.

In order to obtain lower bounds for Dim_λ we relate it with $\mathrm{Dim}_{\mathbf{\Pi},\lambda}$ and to do so we have required an extra condition of regularity over the 'grid' $\mathbf{\Pi} = \{\mathcal{P}_n\}$ (see Section 2). The required condition is trivially fulfilled in the one dimensional case. When the measure λ of a ball is comparable to a power of its diameter we have also obtained an estimate of Dim_λ without assuming the regularity condition on the grid.

Theorem 1.2. *Let (X, d, λ, T) be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$, and let us us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Let $\{r_n\}$ be a non increasing sequence of positive numbers, and let U be an open set in X with $\mu(U) > 0$. Then, for almost all $x_0 \in X$,*

$$\frac{1}{\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{W}(x_0,\{r_n\}) \cap U)} - 1 \leq \frac{\overline{\delta}_\lambda(x_0)\,\overline{\ell}}{h_\mu}\,,$$

where $\overline{\ell} = \limsup_{n\to\infty} \frac{1}{n}\log\frac{1}{r_n}$ and h_μ is the entropy of T with respect to μ.

Moreover, for almost all $x_0 \in X$, the Hausdorff dimensions of the set $\mathcal{W}(U, x_0, \{r_n\})$ verify:

1. *If the grid $\mathbf{\Pi}$ is λ-regular then*

$$\frac{1}{\mathrm{Dim}_\lambda(\mathcal{W}(U,x_0,\{r_n\}))} - 1 \leq \frac{\overline{\delta}_\lambda(x_0)\,\overline{\ell}}{h_\mu}\,.$$

2. *If λ is a doubling measure verifying that $\lambda(B(x,r)) \leq C\, r^s$ for all ball $B(x,r)$, then*

$$\mathrm{Dim}_\lambda(\mathcal{W}(U,x_0,\{r_n\})) \geq 1 - \frac{\overline{\delta}_\lambda(x_0)\overline{\ell}}{s\log\beta}\,.$$

As a consequence, we obtain the same estimates for the Hausdorff dimensions of the set

$$\left\{x \in U:\ \liminf_{n\to\infty}\frac{d(T^n(x),x_0)}{r_n} = 0\right\}.$$

Observe that, for instance, if $X \subset \mathbf{R}$, we obtain that, for any $\alpha > 0$,

$$\mathrm{Dim}_\lambda\{x \in U:\ \liminf_{n\to\infty} e^{n\alpha} d(T^n(x),x_0) = 0\} \geq \frac{h_\mu}{h_\mu + \overline{\delta}_\lambda(x_0)\,\alpha}\,.$$

Theorem 1.2 is sharp since for some expanding systems we get equality, see Theorem 7.4. As in Theorem 1.1 we have chosen to state the above result for almost all $x_0 \in X$ and we refer to Section 6.1 for more precise results concerning to the set of points x_0 where this kind of results holds.

Results related to these two theorems above can be found in [27], [5], [4], [42], [33], [15], [10] and [29]. See also, [13], [17], [7], [43] [16] and [18].

Coding.

It is a well known fact that an expanding map induces a coding on the points of X (see Section 4.1). Via this coding the above results are in certain sense a consequence of analogous results involving symbolic dynamic. More precisely, each point x of the set

$$X_0 := \bigcap_{n=0}^{\infty} \bigcup_{P \in \mathcal{P}_n} P$$

can be codified as $x = [\ i_0\ i_1\ \dots\ ]$ where $P(0, T^n(x)) = P_{i_n} \in \mathcal{P}_0$ for all $n = 0, 1, 2, \dots$ Notice that if $x = [\ i_0\ i_1\ i_2 \dots\ ]$ then $T(x) = [\ i_1\ i_2\ i_3\ \dots\ ]$, i.e. T acts as the left shift on the set of all codes.

Given an increasing sequence $\{t_k\}$ of positive integers and a point $x_0 \in X_0$ we study the size of set

$$\widetilde{\mathcal{W}}(x_0, \{t_n\}) = \{x \in X_0 :\ T^k(x) \in P(t_k, x_0) \text{ for infinitely many } k\}.$$

If $x \in X_0$ and $T^k(x) \in P(t_k, x_0)$ then $P(j, T^k(x)) = P(j, x_0)$ for $j = 0, 1, \dots, t_k$ and it follows that $\widetilde{\mathcal{W}}(x_0, \{t_n\})$ can be also described as the set of points $x = [\ m_0\ m_1 \dots\ ] \in X_0$ such that

$$m_k = i_0,\ m_{k+1} = i_1,\ \dots\ ,\ m_{k+t_k} = i_{t_k}$$

for infinitely many k, where $x_0 = [\ i_0\ i_1 \dots\ ]$. For this set, we have the following analogue of Theorem 1.1:

Theorem 1.3. *Let (X, d, λ, T) be an expanding system. Let x_0 be a point of X_0 such that $\underline{\delta}_\lambda(x_0) > 0$ and let $\{t_n\}$ be a non decreasing sequence of positive integers numbers.*

$$\text{If} \qquad \sum_{n=1}^{\infty} \lambda(P(t_n, x_0)) = \infty, \qquad \text{then} \qquad \lambda(\widetilde{\mathcal{W}}(x_0, \{t_n\})) = \lambda(X).$$

Moreover, if the partition $\mathcal{P}_0$ is finite or if the system has the Bernoulli property, (i.e. if $T(P) = X$ (mod 0) for all $P \in \mathcal{P}_0$), then we have the following quantitative version:

$$\lim_{n\to\infty} \frac{\#\{i \le n :\ T^i(x) \in P(t_i, x_0)\}}{\sum_{j=1}^{n} \mu(P(t_j, x_0))} = 1\,, \qquad \text{for } \lambda\text{-almost every } x \in X\ , \tag{3}$$

where μ is the ACIPM associated to the system.

Property (3) is related to the decay of the correlation coefficients of the indicator functions of $\{P(n, x_0)\}$, see [40] and [35]. For expanding systems with the Bernoulli property L.S. Young [44] has proved that this decay is exponential.

As with $\mathcal{W}(x_0, \{r_n\})$, it is easy to see using the Borel-Cantelli lemma that if $\sum_n \lambda(P(t_n, x_0)) < \infty$, then $\lambda(\widetilde{\mathcal{W}}(x_0, \{t_n\})) = 0$.

Theorem 1.4. *Let (X, d, λ, T) be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is ACIPM associated to the system, and let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Let $\{t_n\}$ be a non decreasing sequence of positive integers and let U be an open set in X with $\mu(U) > 0$. Then, for almost all point $x_0 \in X_0$,*

$$\frac{1}{\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(x_0, \{t_n\}) \cap U)} - 1 \le \frac{\overline{L}(x_0)}{h_\mu}\,,$$

where $\overline{L}(x_0) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda(P(t_n, x_0))}$ and h_μ is the entropy of T with respect to μ. Moreover, if the grid $\mathbf{\Pi}$ is λ-regular, then

$$\frac{1}{\mathrm{Dim}_\lambda(\widetilde{\mathcal{W}}(x_0, \{t_n\}) \cap U)} \le \frac{\overline{L}(x_0)}{h_\mu}\,.$$

As in the previous theorems, for the sake of simplicity, we have stated this last result for almost all point x_0 in X, but we refer to Section 6.1 for a more precise statement concerning the points x_0 which satisfy the conclusions. Also, we should mention that, up a λ-zero measure set, we have that $\overline{L}(x_0)/h_\mu = \limsup_{n\to\infty} t_n/n$ (see Theorem 6.3).

Applications.

The generality of the definition of expanding systems allows us to apply our results in a broad kind of situations. In the final section, we have obtained results for Markov transformations, subshifts of finite type (in particular, Bernoulli shifts), the Gauss transformation, some inner functions and expanding endomorphisms. In the case of Bernoulli shifts we also give a precise upper bound of the dimension by using a large deviation inequality. As an example, for the Gauss map ϕ, which acts on the continued fractions expansions as the left shift, we have the following results:

Theorem 1.5.

(1) *If $\alpha > 1$ then, for almost all $x_0 \in [0,1]$, and more precisely, if x_0 is an irrational number with continued fraction expansion $[\, i_0\,,\ i_1\,,\ \ldots]$ such that $\log i_n = o(n)$ as $n \to \infty$, we have that*

$$\liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = 0\,, \qquad \textit{for almost all } x \in [0,1].$$

(2) *If $\alpha < 1$, then for all $x_0 \in [0,1]$ we have that*

$$\liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = \infty\,, \qquad \textit{for almost all } x \in [0,1].$$

(3) *If x_0 verifies the same hypothesis than in part (1), then*

$$\mathrm{Dim}\left\{x \in [0,1] :\ \liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = 0\right\} = 1\,, \qquad \textit{for any } \alpha > 0.$$

and

$$\mathrm{Dim}\left\{x \in [0,1] :\ \liminf_{n\to\infty} e^{n\kappa}|\phi^n(x) - x_0| = 0\right\} \geq \frac{\pi^2}{\pi^2 + 6\kappa\log 2}\,, \qquad \textit{for any } \kappa > 0.$$

Theorem 1.6. *Let $x_0 \in [0,1]$ be an irrational number with continued fraction expansion $x_0 = [\, i_0, i_1, \ldots]$ and let t_n be a non decreasing sequence of natural numbers. Let $\widetilde{W}$ be the set of points $x = [\, m_0, m_1, \ldots] \in [0,1]$ such that*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \ldots\ ,\ m_{n+t_n} = i_{t_n}\,, \qquad \textit{for infinitely many } n.$$

(1) $\lambda(\widetilde{W}) = 1$, *if*

$$\sum_n \frac{1}{(i_0+1)^2\cdots(i_{t_n}+1)^2} = \infty\,.$$

(2) $\lambda(\widetilde{W}) = 0$, *if*

$$\sum_n \frac{1}{i_0^2\cdots i_{t_n}^2} < \infty\,.$$

(3) *In any case, if $\log i_n = o(n)$ as $n \to \infty$, then*

$$\mathrm{Dim}(\widetilde{W}) \geq \frac{\pi^2}{\pi^2 + 6\log 2\ \limsup_{n\to\infty}\frac{1}{n}\log(i_0+1)^2\cdots(i_{t_n}+1)^2}\,.$$

The techniques developed in this paper for expanding maps and therefore for one-sided Bernoulli shifts, can be extended to bi-sided Bernoulli shifts. This has allowed us [20] to get results on recurrence for Anosov flows.

The outline of the paper is as follows: In Section 2 we give our two definitions of dimension and prove some general results for computing the dimensions of a kind of Cantor-like sets with the particular feature that the ratio between the size of a 'son' and his 'parent' decays very fast. Section 3 contains some consequences of Shannon-McMillan-Breiman Theorem. In section 4 we give the complete definition of an expanding system and in Section 4.1 we recall how to associate a code to the points of X. In Section 4.2 we prove some general properties of expanding maps. The precise statements and proofs of Theorems 1.1 and 1.3, and some consequences of them, are contained in Section 5. The dimension results are included in Section 6. More general versions of Theorems 1.2 and 1.4 are included in Section 6.1. In Section 6.2 we include an upper bound of the dimension. Finally, Section 7 contains several applications of the above results.

Acknowledgements: We want to thank to J. Gonzalo, R. de la Llave, V. Muñoz and R. Pérez Marco for helpful conversations about this work. We are particularly indebted to A. Nicolau for his encouragement and stimulating discusions.

A few words about notation. There are many estimates in this paper involving absolute constants. These are usually denoted by capital letters like C. Occasionally, we shall indicate a constant C depending on some parameter α by $C(\alpha)$. The symbol $\#D$ denotes the number of elements of the set D. By $A \asymp B$ we mean that there exist absolute constants $C_1, C_2 > 0$ such that $C_1 B \leq A \leq C_2 B$.

2 Grids and dimensions.

Along this section $(X, d, \mathcal{A}, \lambda)$ will be a finite measure space with a compatible metric. Compatible means that $\mathcal{A}$ is the the σ-algebra of the Borel sets of d. We recall that we are assuming that the measure λ is non-atomic and its support is X.

Definition 2.1. *Given a set $F \subset X$ and $0 < \alpha \leq 1$, we define the α-dimensional λ-Hausdorff measure of F as*

$$\mathcal{H}^\alpha_\lambda(F) = \lim_{\varepsilon \to 0} \mathcal{H}^\alpha_{\lambda,\,\varepsilon}(F)$$

with

$$\mathcal{H}^\alpha_{\lambda,\,\varepsilon}(F) = \inf \sum_i (\lambda(B_i))^\alpha$$

where the infimum is taken over all the coverings $\{B_i\}$ of F with balls such that $\operatorname{diam}(B_i) \leq \varepsilon$ *for all i.*

It is not difficult to check that $\mathcal{H}^\alpha$ is a regular Borel measure, see e.g. [32]. Observe that if $X \subset \mathbf{R}^N$ and λ is Lebesgue measure, then $\mathcal{H}^\alpha$ is comparable with the usual $N\alpha$-dimensional Hausdorff measure.

Definition 2.2. *The λ-Hausdorff dimension of F is defined as*

$$\operatorname{Dim}_\lambda(F) = \inf\{\alpha : \mathcal{H}^\alpha_\lambda(F) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha_\lambda(F) > 0\}\,.$$

If $X \subset \mathbf{R}^N$ and λ is Lebesgue measure, then the λ-Hausdorff dimension coincides with $1/N$ times the usual Hausdorff dimension.

Definition 2.3. *A grid is a collection $\mathbf{\Pi} = \{\mathcal{P}_n\}$ of partitions of X each of them constituted by disjoint open sets, and such that for all $P_n \in \mathcal{P}_n$ there exists a unique $P_{n-1} \in \mathcal{P}_{n-1}$ such that $P_n \subset P_{n-1}$, and $\sup_{P \in \mathcal{P}_n} \operatorname{diam}(P) \to 0$ as $n \to \infty$..*

Definition 2.4. *Given a grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$ of X and $0 < \alpha \leq 1$, the α-dimensional λ-grid measure of any subset $F \subset X$ is defined as*

$$\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(F) = \lim_{n \to \infty} \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda,n}(F)$$

with

$$\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda,n}(F) = \inf \sum_i (\lambda(P_i))^\alpha$$

where the infimum is taken over all the coverings $\{P_i\}$ of F with sets $P_i \in \cup_{k \geq n} \mathcal{P}_k$.

The λ-grid Hausdorff dimension of F is defined as

$$\operatorname{Dim}_{\mathbf{\Pi},\lambda}(F) = \inf\{\alpha : \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(F) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(F) > 0\}\,.$$

As before we have that $\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}$ is a Borel measure.

Remark 2.1. If $X \subseteq \mathbf{R}$ and λ is Lebesgue measure we have that $\mathcal{H}^\alpha_\lambda(F) \leq \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(F)$ and therefore, for any $F \subset \mathbf{R}$,

$$\operatorname{Dim}_\lambda(F) \leq \operatorname{Dim}_{\mathbf{\Pi},\lambda}(F)\,.$$

Also, if $X = [0,1]$, $\mathcal{P}_n$ denotes the family of dyadic intervals with length $1/2^{n+1}$ and λ is Lebesgue measure, then we have, for any $F \subset [0,1]$, that

$$\operatorname{Dim}_\lambda(F) = \operatorname{Dim}_{\mathbf{\Pi},\lambda}(F)\,.$$

In order to compute the λ-grid Hausdorff dimension we will use the following result which parallels Frostman lemma.

Lemma 2.1. *Let $\mathbf{\Pi} = \{\mathcal{P}_n\}$ be a grid of X. For each $n \in \mathbf{N}$, let $\mathcal{Q}_n$ be a subcollection of $\mathcal{P}_n$ and let F be a set with*

$$F \subseteq \bigcap_n \bigcup_{Q \in \mathcal{Q}_n} Q \,.$$

If there exist a measure ν such that $\nu(F) > 0$, a real number $0 < \gamma \leq 1$ and a positive constant C such that, for all $x \in F$,

$$\nu(Q(k,x)) \leq C\left(\lambda(Q(k,x))\right)^{\gamma},$$

where $Q(k,x)$ denotes the block of $\mathcal{Q}_k$ which contains x, then,

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(F) \geq \gamma \,.$$

Proof. It follows from the fact that $\mathcal{H}^{\alpha}_{\mathbf{\Pi},\lambda}(F) \geq \nu(F)/C > 0$. □

The following result allows to obtain a lower bound for the λ-grid Hausdorff dimension of Cantor-like sets.

Theorem 2.1. *Let $\mathbf{\Pi} = \{\mathcal{P}_n\}$ be a grid of X and let $\{d_j\}$ and $\{\widetilde{d}_j\}$ be two increasing sequences of natural numbers tending to infinity verifying that $d_{j-1} < \widetilde{d}_j < d_j$ for each j.*

Consider two collections $\{\mathcal{J}_j\}$ and $\{\widetilde{\mathcal{J}}_j\}$ of subsets of X such that:

- *(i)* $\widetilde{\mathcal{J}}_0 = \mathcal{J}_0 = \{J_0\}$ *and for each j, $\mathcal{J}_j \subseteq \mathcal{P}_{d_j}$ and $\widetilde{\mathcal{J}}_j \subseteq \mathcal{P}_{\widetilde{d}_j}$.*
- *(ii) For each $J_j \in \mathcal{J}_j$ ($j \geq 1$) there exists a unique $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ such that* $\mathrm{closure}(J_j) \subset \widetilde{J}_j$. *Reciprocally, for each $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ there exists a unique $J_j \in \widetilde{\mathcal{J}}_j$ such that* $\mathrm{closure}(J_j) \subset \widetilde{J}_j$.
- *(iii) For each $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ ($j \geq 1$) there exists a unique $J_{j-1} \in \mathcal{J}_{j-1}$ such that $\widetilde{J}_j \subset J_{j-1}$.*

Let $\mathcal{C}$ be the Cantor-like set defined by

$$\mathcal{C} = \bigcap_{j=0}^{\infty} \bigcup_{J_j \in \mathcal{J}_j} J_j = \bigcap_{j=0}^{\infty} \bigcup_{\widetilde{J}_j \in \widetilde{\mathcal{J}}_j} \widetilde{J}_j \,.$$

Assume that the pattern of $\mathcal{C}$ has the following additional properties:

1. *There exist two sequences $\{\alpha_j\}$ and $\{\beta_j\}$ of positive numbers such that*

$$\alpha_j \leq \frac{\lambda(\widetilde{J}_j)}{\lambda(J_{j-1})} \leq \beta_j \,.$$

2. *There exists a sequence $\{\gamma_j\}$ of positive numbers such that*

$$\frac{\lambda(J_j)}{\lambda(\widetilde{J}_j)} \geq \gamma_j \,.$$

3. *There exists a sequence $\{\delta_j\}$ with $0 < \delta_j \leq 1$ such that*

$$\lambda(\widetilde{J}_j \cap J_{j-1}) \geq \delta_j \, \lambda(J_{j-1}) \,.$$

4. *There exists an absolute constant Λ such that for all j large enough*

$$\frac{1}{\delta_{j+1}} \frac{\beta_j}{\delta_j} \frac{\beta_{j-1}}{\delta_{j-1}} \cdots \frac{\beta_1}{\delta_1} \leq \left[(\alpha_j\gamma_j)\,(\alpha_{j-1}\gamma_{j-1}) \,\cdots\, (\alpha_1\gamma_1)\right]^{\Lambda} .$$

Then

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{C}) \geq \Lambda \,.$$

Remark 2.2. Observe that in the special case when the two families $\widetilde{\mathcal{J}}_j$ and $\mathcal{J}_j$ coincide and $\alpha_j = \alpha$, $\beta_j = \beta$, $\delta_j = \delta$, then the above result is the usual Hungerford's Lemma ($\Lambda = \log(\beta/\delta)/\log\alpha$), see e.g. [37].

Proof. We construct a probability measure ν supported on $\mathcal{C}$ in the following way: We define $\nu(J_0) = 1$ and for each set $J_j \in \mathcal{J}_j$ we write

$$\nu(J_j) = \nu(\widetilde{J}_j) = \frac{\lambda(\widetilde{J}_j)}{\lambda(\widetilde{\mathcal{J}}_j \cap J_{j-1})}\, \nu(J_{j-1})$$

where J_{j-1} and $\widetilde{J}_j$ denote the unique sets in $\mathcal{J}_{j-1}$ and $\widetilde{\mathcal{J}}_j$ respectively, such that $J_j \subset \widetilde{J}_j \subset J_{j-1}$. As usual, for any Borel set B, the ν-measure of B is defined by

$$\nu(B) = \nu(B \cap \mathcal{C}) = \inf \sum_{U \in \mathcal{U}} \nu(U)$$

where the infimum is taken over all the coverings $\mathcal{U}$ of $B \cap \mathcal{C}$ with sets in $\bigcup \mathcal{J}_j$.

We will show that there exists a positive constant C such that for all $x \in \mathcal{C}$ and m large enough,

$$\nu(P(m,x)) \leq C\, (\lambda(P(m,x))^{\Lambda} \tag{4}$$

and therefore, from Lemma 2.1, we get the result.

To prove (4) let us suppose first that $P(m,x) = J_j$ for some $J_j \in \mathcal{J}_j$. From properties (1)-(3) we have that

$$\begin{aligned}
&\nu(J_j) \leq \frac{\beta_j}{\delta_j}\, \nu(J_{j-1}) \leq \frac{\beta_j}{\delta_j} \frac{\beta_{j-1}}{\delta_{j-1}} \cdots \frac{\beta_1}{\delta_1}\,,\\
&\lambda(J_j) \geq \alpha_j \gamma_j\, \lambda(J_{j-1}) \geq (\alpha_j \gamma_j)\, (\alpha_{j-1}\gamma_{j-1}) \,\cdots\, (\alpha_1 \gamma_1) \lambda(J_0)\,.
\end{aligned}$$

and follows from property (4) that

$$\nu(\widetilde{J}_j) = \nu(J_j) \leq C\, \delta_{j+1} \lambda(J_j)^{\Lambda}\,. \tag{5}$$

This condition is stronger than (4) for $\widetilde{J}_j$ and J_j and we will use it to get (4) in general.

Now, let us suppose that $P(m,x) \neq J_j$ for all j and for all $J_j \in \mathcal{J}_j$. Since $x \in \mathcal{C}$ there exist $J_j \in \mathcal{J}_j$ and $J_{j+1} \in \mathcal{J}_{j+1}$ such that

$$J_{j+1} \subset P(m,x) \subset J_j\,.$$

If $P(m,x) \subset \widetilde{J}_{j+1}$, then from the definition of ν and (5) for $\widetilde{J}_{j+1}$ we get

$$\nu(P(m,x)) = \nu(\widetilde{J}_{j+1}) = \nu(J_{j+1}) \leq C\, (\lambda(J_{j+1}))^{\Lambda} \leq C\, (\lambda(P(m,x)))^{\Lambda}\,.$$

Otherwise $P(m,x)$ contains sets of the family $\widetilde{\mathcal{J}}_{j+1}$ and we have that

$$\begin{aligned}
\nu(P(m,x)) &= \sum_{\substack{\tilde{J}_{j+1} \in \widetilde{\mathcal{J}}_{j+1}\\ \tilde{J}_{j+1} \subseteq P(m,x)}} \nu(J_{j+1}) = \sum_{\substack{\tilde{J}_{j+1} \in \widetilde{\mathcal{J}}_{j+1}\\ \tilde{J}_{j+1} \subseteq P(m,x)}} \frac{\lambda(\widetilde{J}_{j+1})}{\lambda(\widetilde{\mathcal{J}}_{j+1} \cap J_j)}\, \nu(J_j)\\
&= \frac{\nu(J_j)}{\lambda(\widetilde{\mathcal{J}}_{j+1} \cap J_j)} \sum_{\substack{\tilde{J}_{j+1} \in \widetilde{\mathcal{J}}_{j+1}\\ \tilde{J}_{j+1} \subseteq P(m,x)}} \lambda(\widetilde{J}_{j+1}) \leq \frac{\nu(J_j)}{\lambda(\widetilde{\mathcal{J}}_{j+1} \cap J_j)}\, \lambda(P(m,x))\,.
\end{aligned} \tag{6}$$

And using property (3) and (5) we obtain that

$$\nu(P(m,x)) \leq \frac{C}{(\lambda(J_j))^{1-\Lambda}}\, \lambda(P(m,x))$$

But $\lambda(J_j)) \geq \lambda(P(m,x))$ and so we get

$$\nu(P(m,x)) \leq C\, (\lambda(P(m,x)))^{\Lambda}\,.$$

$\square$

Remark 2.3. Notice that if we define

$$\nu(J_j) = \frac{\lambda(J_j)}{\lambda(\mathcal{J}_j \cap J_{j-1})}\, \nu(J_{j-1})$$

then instead of (6) we get

$$\begin{aligned}\nu(P(m,x)) &\le \frac{\nu(J_j)}{\lambda(\mathcal{J}_{j+1} \cap J_j)} \sum_{\substack{\tilde{J}_{j+1} \in \tilde{\mathcal{J}}_{j+1} \\ \tilde{J}_{j+1} \subseteq P(m,x)}} \lambda(J_{j+1}) \\ &\le \frac{\nu(J_j)\,\omega_{j+1}}{\lambda(\mathcal{J}_{j+1} \cap J_j)} \sum_{\substack{\tilde{J}_{j+1} \in \tilde{\mathcal{J}}_{j+1} \\ \tilde{J}_{j+1} \subseteq P(m,x)}} \lambda(\tilde{J}_{j+1}) \le \frac{1}{\delta_{j+1}} \frac{\nu(J_j)}{\lambda(J_j)} \frac{\omega_{j+1}}{\gamma_{j+1}}\, \lambda(P(m,x))\,.\end{aligned}$$

where

$$\gamma_j \le \frac{\lambda(J_j)}{\lambda(\tilde{J}_j)} \le \omega_j\,.$$

Hence if $\nu(J_j) \le C\delta_{j+1}(\lambda(J_j))^{\Lambda}$ we get that

$$\nu(P(m,x)) \le \frac{C}{(\lambda(J_j))^{1-\Lambda}} \frac{\omega_{j+1}}{\gamma_{j+1}}\, \lambda(P(m,x))$$

and we will need

$$\frac{\omega_{j+1}}{\gamma_{j+1}} \le \frac{1}{(\lambda(P(m,x)))^{\varepsilon}}$$

in order to get that the dimension is greater than $\Lambda - \varepsilon$. We recall that in this case the upper bound for $\lambda(P(m,x))$ is $\lambda(J_j)$ and $\lambda(J_j) \le (\omega_j\beta_j)(\omega_{j-1}\beta_{j-1})\cdots(\omega_1\beta_1)$.

Corollary 2.1. *Under the same hypotheses that in Theorem 2.1 we have that if $\delta_j = \delta > 0$ and*

$$\alpha_j = e^{-N_j a}\,, \qquad \beta_j = e^{-N_j b}\,, \qquad \gamma_j = e^{-N_j c}\,,$$

then

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{C}) \ge \frac{b}{a+c} - \frac{\log(1/\delta)}{a+c} \lim_{j\to\infty} \frac{j}{N_1 + \cdots + N_j}\,.$$

The next definition states some kind of regularity on the distribution of the blocks of the partitions. This property will allow us to relate the Hausdorff dimension with the grid Hausdorff dimension.

Definition 2.5. *Let $\mathbf{\Pi} = \{\mathcal{P}_n\}$ be a grid of X. We will say that $\mathbf{\Pi}$ is λ-regular if there exists a positive constant C such that for all ball B*

$$\lambda(\cup\{P : P \in \mathcal{P}_n\,,\, P \cap B \ne \emptyset\}) \le C\,\lambda(B)$$

for all n such that $\sup_{P\in\mathcal{P}_n} \lambda(P) \le \lambda(B)$.

Remark 2.4. It is clear from the definition that any grid of $X \subset \mathbf{R}$ is λ-regular (we can take $C = 3$)

An example: Let X be the square $[0,1] \times [0,1]$ in $\mathbf{R}^2$ and let us denote by λ the Lebesgue measure. Consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$ defined as follows: the elements of $\mathcal{P}_0$ are the four open rectangles obtained by dividing the square $[0,1] \times [0,1]$ through the lines $x = a$ and $y = b$, with $\frac{1}{2} < b < a < 1$; the elements of $\mathcal{P}_n$ are getting by dividing each rectangle of $\mathcal{P}_{n-1}$ in four rectangles using the same proportions. We will see that this is not a regular grid.

Let us consider the ball B_k with diameter $(1-b)^k$ and contained in the square $[0,(1-b)^k] \times [(1-b)^k, 1]$. It is easy to see that $\sup_{P\in\mathcal{P}_n} \lambda(P) = (ab)^n$, and therefore $(ab)^n \le \lambda(B_k)$ implies

$$n \ge \frac{\log c + 2k \log(1/(1-b))}{\log 1/(ab)}$$

Therefore, if $\mathbf{\Pi}$ is regular, then for $n = n(k) = 2k\frac{\log(1/(1-b))}{\log 1/(ab)} + C$ the quotient

$$C_k := \frac{\lambda(\cup\{P : P \in \mathcal{P}_n\,,\, P \cap B_k \ne \emptyset\})}{\lambda(B_k)}$$

has to be bounded. But, it is easy to see that the elements of $\mathcal{P}_n$ whose closure intersects to $[0,1] \times \{0\}$ are rectangles of width a^n, and hence, since $b < a$,

$$C_k \geq \frac{(1-b)^k a^{n(k)}}{(1-b)^{2k}} \to \infty \qquad \text{as } k \to \infty.$$

Therefore, this grid is not regular. On the other hand it is clear that any grid in X whose elements are all squares is regular.

The following result gives a lower bound for the Hausdorff dimension of Cantor like sets which are constructed using a regular subgrid with some control into the quotient between the size of parents and sons.

Proposition 2.1. *Let $\mathbf{\Pi} = \{\mathcal{P}_n\}$ be a grid of X and let $\{\mathcal{Q}_n\}$ be a λ-regular subgrid of $\mathbf{\Pi}$. Let us suppose that there exist strictly non increasing sequences $\{a_n\}$, $\{b_n\}$ of positive numbers such that $\lim_{n\to\infty} b_n = 0$ and for all $Q \in \mathcal{Q}_n$*

$$a_n \leq \lambda(Q) \leq b_n\,.$$

Then, for any subset $F \subseteq \bigcap_n \bigcup_{Q\in\mathcal{Q}_n} Q$,

$$\mathcal{H}^{\alpha}_{\mathbf{\Pi},\lambda}(F) \leq C\,\mathcal{H}^{1-(1-\alpha)\eta}_{\lambda}(F) \tag{7}$$

for all α and η such that

$$\limsup_{n\to\infty} \frac{\log(1/a_n)}{\log(1/b_{n-1})} < \eta < \frac{1}{1-\alpha}\,, \tag{8}$$

where C is an absolute positive constant. In particular,

$$\frac{1-\mathrm{Dim}_\lambda(F)}{1-\mathrm{Dim}_{\mathbf{\Pi},\lambda}(F)} \leq \limsup_{n\to\infty} \frac{\log(1/a_n)}{\log(1/b_{n-1})}\,. \tag{9}$$

Proof. Let us consider a ball B such that $B \cap F \neq \emptyset$ and let $n = n(B)$ be the smallest integer such that $b_n \leq \lambda(B)$. Then

$$b_n \leq \lambda(B) < b_{n-1}.$$

We denote by $\mathcal{Q}(B)$ the collection of elements in $\mathcal{Q}_n$ whose intersection with B is not empty. Then the collection $\mathcal{Q}(B)$ is a covering of $B \cap F$, that is

$$B \cap F \subset \bigcup\{Q \,:\, Q \in \mathcal{Q}(B)\}, \tag{10}$$

and moreover by the Definition 2.5 and the election of $n = n(B)$,

$$\sum_{Q\in\mathcal{Q}(B)} (\lambda(Q))^\alpha = \sum_{Q\in\mathcal{Q}(B)} \frac{1}{(\lambda(Q))^{1-\alpha}}\,\lambda(Q) \leq C\,\frac{1}{a_n^{1-\alpha}}\,\lambda(B)\,.$$

We may assume that n is large because $\mathrm{diam}\,(B)$ is small and so the above inequality and (8) imply that

$$\sum_{Q\in\mathcal{Q}(B)} (\lambda(Q))^\alpha \leq C'(\lambda(B))^{1-(1-\alpha)\eta}\,. \tag{11}$$

The inequality (7) follows now from (10) and (11). To prove (9) let us observe that we can assume that

$$\limsup_{n\to\infty} \frac{\log(1/a_n)}{\log(1/b_{n-1})} < \frac{1}{1-\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathrm{F})}$$

since in other case (9) is trivial. Let us choose now α and η such that

$$\limsup_{n\to\infty} \frac{\log(b_{n-1}/a_n)}{\log(1/b_n)} < \eta < \frac{1}{1-\alpha} < \frac{1}{1-\mathrm{Dim}_{\mathbf{\Pi},\lambda}(F)}. \tag{12}$$

Then $\mathcal{H}^{\alpha}_{\mathbf{\Pi},\lambda}(F) > 0$ and by (7) we have also that $\mathcal{H}^{1-(1-\alpha)\eta}_{\lambda}(F) > 0$. Since α y η are arbitrary numbers verifying (12), the ineguality (9) follows. $\square$

3 Some consequences of Shannon-McMillan-Breiman Theorem

Along this section $(X, \mathcal{A}, \mu)$ will be a finite measure space and $T : X \longrightarrow X$ will be a measurable transformation. A *partition* of X is a family $\mathcal{P}$ of measurable sets with positive measure satisfying

1. If $P_1, P_2 \in \mathcal{P}$ then $\mu(P_1 \cap P_2) = 0$.
2. $\mu\left(X \setminus \cup_{P \in \mathcal{P}} P\right) = 0$.

It follows from these properties that $\mathcal{P}$ must be finite or numerable. The *entropy* of a partition $\mathcal{P}$ is defined as

$$H_\mu(\mathcal{P}) = \sum_{P \in \mathcal{P}} \mu(P) \, \log \frac{1}{\mu(P)} \, .$$

If $T : X \longrightarrow X$ preserves the measure μ, then the *entropy of* T *with respect to the partition* $\mathcal{P}$ is

$$h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} \, H_\mu\big(\vee_{j=0}^{n-1} \, T^{-j}\mathcal{P}\big).$$

This limit exists since the sequence in the right hand side is decreasing. Hence $h_\mu(T, \mathcal{P}) \leq H_\mu(\mathcal{P})$.

Finally, the *entropy* $h_\mu(T)$ of the endomorphism T is the supremum of $h_\mu(T, \mathcal{P})$ over all the partitions $\mathcal{P}$ of X with entropy $h_\mu(T, \mathcal{P}) < \infty$.

If the partition $\mathcal{P}$ is generating, i.e. if $\vee_{j=0}^{\infty} T^{-j}(\mathcal{P})$ generates $\mathcal{A}$, then, by the Kolmogorov-Sinai Theorem ([M, p. 218-220]), we get

Theorem C *Let* $(X, \mathcal{A}, \mu)$ *be a probability space and* $T : X \longrightarrow X$ *be a measure preserving transformation. If* $\mathcal{P}$ *is a generating partition of* X *and the entropy* $H_\mu(\mathcal{P})$ *is finite, then* $h_\mu(T) = h_\mu(T, \mathcal{P})$.

Let $P(n, x)$ denotes the element of the partition $\vee_{j=0}^{n} T^{-j}(\mathcal{P})$ which contains the point $x \in X$. It follows from the definition of partition, that for almost every $x \in X$, $P(n, x)$ is defined for all n. Entropy is a measure of how fast $\mu(P(n, x))$ goes to zero. The following fundamental result, which is due to Shannon, McMillan and Breiman, formalizes this assertion:

Theorem D([M, p. 209]) *Let* $(X, \mathcal{A}, \mu)$ *be a probability space and let* $T : X \longrightarrow X$ *be a measure preserving ergodic transformation. Let* $\mathcal{P}$ *be a partition with finite entropy* $H_\mu(\mathcal{P})$*. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu(P(n, x))} = h_\mu(T, \mathcal{P}) \, ,$$

for μ*-almost every* $x \in X$.

We will need later the following consequence of Theorem D.

Lemma 3.1. *Let* $(X, \mathcal{A}, \mu)$ *be a probability space and let* $T : X \longrightarrow X$ *be a measure preserving ergodic transformation and* $\mathcal{P}$ *be a partition with finite entropy* $H_\mu(\mathcal{P})$*. Then, given* $\varepsilon > 0$ *there exists a decreasing sequence of sets* $\{E_N^\varepsilon\}_{N\in\mathbf{N}}$ *such that*

$$\mu(E_N^\varepsilon) \to 0 \qquad \textit{as} \qquad N \to \infty \tag{13}$$

and for all $x \in X \setminus E_N^\varepsilon$

$$e^{-j(h_\mu+\varepsilon)} < \mu(P(j, x)) < e^{-j(h_\mu-\varepsilon)} \, , \qquad \textit{for all } j \geq N \, . \tag{14}$$

with $h_\mu = h_\mu(T, \mathcal{P})$ *the entropy of* T *with respect to* μ *and the partition* $\mathcal{P}$.

Proof. Given $\varepsilon > 0$ we define for all $j \in \mathbf{N}$ the sets

$$F_j^\varepsilon = \left\{ x \in X \, : \, \left| \frac{1}{j} \log \frac{1}{\mu(P(j, x))} - h_\mu \right| < \varepsilon \right\} \, .$$

By Theorem D we know that for almost every $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu(P(n, x))} = h_\mu \, .$$

Therefore there is a set S with $\mu(S)=0$ such that for all $x \in X \setminus S$ there exists $n(x) \in \mathbf{N}$ such that

$$\left| \frac{1}{j} \log \frac{1}{\mu(P(j,x))} - h_\mu \right| < \varepsilon \qquad \text{for all } j \geq n(x)\,.$$

Hence,

$$X \setminus S \subset \bigcup_{N \in \mathbf{N}} \bigcap_{j \geq N} F_j^\varepsilon$$

or equivalently

$$\bigcap_{N \in \mathbf{N}} \bigcup_{j \geq N} (X \setminus F_j^\varepsilon) \subset S\,. \tag{15}$$

We define

$$E_N^\varepsilon = \bigcup_{j \geq N} (X \setminus F_j^\varepsilon)\,.$$

By definition $E_{N+1}^\varepsilon \subset E_N^\varepsilon$ for all $N \in \mathbf{N}$ and by (15) $\mu(\bigcap_N E_N^\varepsilon)=0$, therefore

$$\mu(E_N^\varepsilon) \to 0 \qquad \text{when} \qquad N \to \infty\,.$$

Moreover, if $x \in X \setminus E_N^\varepsilon$, then $x \in F_j^\varepsilon$ for all $j \geq N$, and therefore

$$e^{-j(h_\mu+\varepsilon)} < \mu(P(j,x)) < e^{-j(h_\mu-\varepsilon)} \qquad \text{for all} \qquad j \geq N\,.$$

$\square$

Proposition 3.1. *Let $(X,\mathcal{A},\mu)$ be a probability space, let $T: X \longrightarrow X$ be a measure preserving mixing transformation, and $\mathcal{P}$ be a partition with finite entropy $H_\mu(\mathcal{P})$. Let us denote*

$$X_0 = \bigcap_{n=0}^{\infty} \bigcup_{P \in \vee_{j=0}^n T^{-j}(\mathcal{P})} P\,.$$

Let P_1, P_2 be two fixed elements of $\mathcal{P}$. For $\varepsilon > 0$ let $\{E_M^\varepsilon\}$ be the decreasing sequence of sets given by Lemma 3.1 . If $\mathcal{S}_{N,M}$ denotes the collection of the sets $P(N,x)$ verifying

$$x \in X_0 \setminus E_M^\varepsilon\,, \qquad P(N,x) \subset P_1\,, \qquad T^N(P(N,x)) = P(0,T^N(x)) = P_2\,,$$

then, for all M and N large enough depending on P_1 and P_2,

$$\mu(\mathcal{S}_{N,M}) := \mu\Big(\bigcup_{S \in \mathcal{S}_{N,M}} S\Big) \geq \frac{1}{2}\,\mu(P_1)\,\mu(P_2)\,. \tag{16}$$

Proof. We have that

$$\begin{aligned}
\mu(P_1) &= \mu(\mathcal{S}_{N,M}) + \sum_{P \in \mathcal{P}\setminus\{P_2\}} \sum_{\substack{P(N,x) \ \ s.t. \ \ x \in X_0 \setminus E_M^\varepsilon \\ T^N(P(N,x))=P}} \mu(P(N,x)) + \mu(P_1 \cap E_M^\varepsilon) \\
&\leq \mu(\mathcal{S}_{N,M}) + \sum_{P \in \mathcal{P}\setminus\{P_2\}} \mu(P_1 \cap T^{-N}(P)) + \mu(P_1 \cap E_M^\varepsilon) \\
&= \mu(\mathcal{S}_{N,M}) + \mu(P_1) - \mu(P_1 \cap T^{-N}(P_2)) + \mu(P_1 \cap E_M^\varepsilon)\,.
\end{aligned}$$

Notice that $\lim_{M\to\infty} \mu(E_M^\varepsilon) = 0$ by Lemma 3.1 and

$$\lim_{N\to\infty} \mu(P_1 \cap T^{-N}(P_2)) = \mu(P_1)\,\mu(P_2)$$

because T is mixing. Hence, for M and N large enough,

$$\mu(\mathcal{S}_{N,M}) \geq \frac{1}{2}\,\mu(P_1)\,\mu(P_2)\,.$$

$\square$

4 Expanding maps.

We will say that $(X, d, \mathcal{A}, \lambda, T)$ is an *expanding system* if $(X, \mathcal{A}, \lambda)$ is a finite measure space, λ is a non-atomic measure and the support of λ is equal to X, (X, d) is a locally complete separable metric space, $\mathcal{A}$ is its Borel σ-algebra and $T : X \longrightarrow X$ is an *expanding map*, i.e. a measurable transformation satisfying the following properties:

(A) There exists a collection of open sets $\mathcal{P}_0 = \{P_i\}$ of X such that $\sup\limits_{P \in \mathcal{P}_0} \operatorname{diam}(P) < \infty$, and

 (1) $\lambda(P_i) > 0$,
 (2) $P_i \cap P_j = \emptyset$ if $i \neq j$,
 (3) $\lambda(X \setminus \cup_i P_i) = 0$,
 (4) The restriction $T\big|_{P_i}$ of T to the set P_i is injective,
 (5) For each P_i, if $P_j \cap T(P_i) \neq \emptyset$, then $P_j \subseteq T(P_i)$.
 (6) For each P_i, if $P_j \subseteq T(P_i)$, then the map $T\big|_{P_i}^{-1} : T(P_i) \cap P_j \longrightarrow P_i$ is open.
 (7) There is a natural number $n_0 > 0$ such that $\lambda(T^{-n_0}(P_i) \cap P_j) > 0$, for all $P_i,\ P_j \in \mathcal{P}_0$.

(B) There exists a measurable map $\mathbf{J} : X \longrightarrow [0, \infty)$, $\mathbf{J} > 0$ in $\cup_{P \in \mathcal{P}_0} P$, such that for all $P_i \in \mathcal{P}_0$ and for all Borel subset A of P_i we have that

$$\lambda(T(A)) = \int_A \mathbf{J}\, d\lambda\,.$$

and moreover there exist absolute constants $0 < \alpha \leq 1$ and $C_1 > 0$, such that for all $x, y \in P_i$

$$\left| \frac{\mathbf{J}(x)}{\mathbf{J}(y)} - 1 \right| \leq C_1\, d(T(x), T(y))^{\alpha}\,.$$

(C) Let us define inductively the following collections $\{\mathcal{P}_i\}$ of open sets:

$$\mathcal{P}_1 = \bigcup_{P_i \in \mathcal{P}_0} \{(T\big|_{P_i})^{-1}(P_j)\ :\ P_j \in \mathcal{P}_0\,,\ P_j \subset T(P_i)\},$$

and, in general,

$$\mathcal{P}_n = \bigcup_{P_i \in \mathcal{P}_0} \{(T\big|_{P_i})^{-1}(P_j)\ :\ P_j \in \mathcal{P}_{n-1}\,,\ P_j \subset T(P_i)\}.$$

Then, there exist absolute constants $\beta > 1$ and $C_2 > 0$ such that for all x, y in the same element of $\mathcal{P}_n$ we have that

$$d(T^n(x), T^n(y)) \geq C_2 \beta^n d(x, y)\,.$$

Remark 4.1.

1. It is easy to see that each family $\mathcal{P}_n$ verifies the properties (A.1), (A.2) and (A.3). Also notice that, for each n, $T(\mathcal{P}_n)$ is equal to $\mathcal{P}_{n-1}$ (mod 0) in the sense that the image of each element of $\mathcal{P}_n$ is an element of $\mathcal{P}_{n-1}$ (mod 0).
2. From the properties (A.1), (A.2) and (A.3) it follows that $\mathcal{P}_0$ is finite or numerable.
3. As a consequence of property (C) we have, since $\sup_{P \in \mathcal{P}_0} \operatorname{diam}(P) < \infty$, that

$$\lim_{n \to \infty} \Big(\sup_{P \in \mathcal{P}_n} \operatorname{diam}(P) \Big) = 0\,.$$

 Therefore (see for example [M, p.13]) we deduce that the partition $\mathcal{P}_0$ is *generating*. This means that $\mathcal{A} = \bigvee_{n=0}^{\infty} \mathcal{P}_n$ (mod 0).

We will define also, for each $n \in \mathbf{N}$, the function

$$\mathbf{J}_n(x) = \mathbf{J}(x) \cdot \mathbf{J}(T(x)) \cdots \mathbf{J}(T^{n-1}(x))\,, \qquad \text{for } x \in \bigcup_{P \in \mathcal{P}_{n-1}} P.$$

Then it follows easily that

$$\int_A f(T^n(x))\,\mathbf{J}_n(x)\,d\lambda(x) = \int_{T^n(A)} f(x)\,d\lambda(x)\,, \qquad \text{for all } f\in L^1(\mu)\,, \tag{17}$$

and, in particular,

$$\lambda(T^n(A)) = \int_A \mathbf{J}_n\,d\lambda$$

for each measurable set A contained in some element of $\mathcal{P}_{n-1}$.

Notice that in the definition of an expanding map the measure λ it is not required to have special dynamical properties. However it is a remarkable fact that it is possible to find an invariant measure which is essentially comparable to λ and has very interesting dynamical properties. More concretely it is known the following result

Theorem E ([M, p.172]). *Let $(X,d,\mathcal{A},\lambda,T)$ be an expanding system. Then, there exist a unique probability measure μ on $\mathcal{A}$ which is absolutely continuous with respect to λ and such that*

(i) *T preserves the measure μ.*

(ii) *$d\mu/d\lambda$ is Hölder continuous.*

(iii) *For each $P_i\in\mathcal{P}_0$ there exist a positive constant K_i such that*

$$\frac{1}{K_i}\le\frac{d\mu}{d\lambda}(x)\le K_i\,, \qquad \textit{for all } x\in P_i\,.$$

(iv) *T is exact with respect to μ.*

(v) *$\mu(B)=\dfrac{1}{\lambda(X)}\lim_{n\to\infty}\lambda(T^{-n}(B))$ for every $B\in\mathcal{A}$.*

In what follows we will refer to μ as the ACIPM measure associated to the expanding system.

Remark 4.2. Notice that by part (iii) and property (A.3) of expanding maps the measures λ and μ have the same zero measure sets and therefore the same full measure sets.

Remark 4.3. We recall that the condition (A.4) in the definition of expanding maps says that $T\big|P_i$ must be injective for all $P_i\in\mathcal{P}_0$. If we strengthen this condition by requiring also that

$$\inf_{P\in\mathcal{P}_0}\lambda(T(P))>0 \qquad\text{and}\qquad \sup_{P\in\mathcal{P}_0}\operatorname{diam}(T(P))<\infty\,,$$

or, in particular, if $T:P\longrightarrow X$ is bijective (mod 0) for all $P\in\mathcal{P}_0$ and X is bounded, then, a slight modification of the proof of Theorem E in [30] (using Remark 4.6 instead of [M, Lemma 1.5]), allows to obtain the property (iii) of μ with an absolute constant K. Therefore with this additional assumption one have that

$$\frac{1}{K}\,\lambda(A)\le\mu(A)\le K\,\lambda(A)\,, \qquad \text{for all } A\in\mathcal{A}\,.$$

Of course, this condition also holds if the partition $\mathcal{P}_0$ is finite.

We recall that since $\sup_{P\in\mathcal{P}_0}\operatorname{diam}(P)<\infty$ then the partition $\mathcal{P}_0$ is generating, Therefore, for expanding systems $h_\mu(T)=h_\mu(T,\mathcal{P}_0)$.

Remark 4.4. By definition of entropy, if $H_\mu(\mathcal{P}_0)$ is finite, then $h_\mu(T)\le H_\mu(\mathcal{P}_0)<\infty$. Also, since T is exact with respect to μ we have that an expanding system is Kolmogorov ([30], p. 158). Also, X is a Lebesgue space ([30], p. 81). As a consequence it follows that $h_\mu(T)>0$, ([30]., p. 225).

For expanding maps there exists an alternative way of computing the entropy of T:

Theorem F([M, p. 227]) *Let $(X,d,\mathcal{A},\lambda,T)$ be an expanding system and let μ be the ACIPM measure associated to the system. If the entropy $H_\mu(\mathcal{P}_0)$ of the partition $\mathcal{P}_0$ is finite, then* $\log\mathbf{J}$ *is integrable and*

$$h_\mu(T)=\int_X\log\mathbf{J}\,d\mu\,.$$

4.1 A code for expanding maps

We will denote by $P(n,x)$ the element of the collection $\mathcal{P}_n$ which contains the point $x \in X$. Observe that for each n, $P(n,x)$ is well defined for x belonging to

$$\Upsilon_n := \cup\{P : P \in \mathcal{P}_n\}$$

and Υ_n has full λ-measure for property (A.3) for $\mathcal{P}_n$, see Remark 4.1.1. Therefore if x belongs to the set

$$X_0 := \bigcap_{n=0}^{\infty} \Upsilon_n = \bigcap_{n=0}^{\infty} \bigcup_{P \in \mathcal{P}_n} P \tag{18}$$

then $P(n,x)$ is well defined for all n. Moreover, if $x \in \Upsilon_n$ then from the definition of $\mathcal{P}_n$ we have that $T(x) \in \Upsilon_{n-1}$. Hence, if $x \in X_0$ we have that $T^\ell(x) \in X_0$ for all $\ell \in \mathbf{N}$, and so $P(n, T^\ell(x))$ is well defined for all $n, \ell \in \mathbf{N}$. This set has full λ-measure since $X \setminus X_0 \subseteq \cup_{n \geq 0} X \setminus \Upsilon_n$ and this set has zero λ-measure by (A.3) for $\mathcal{P}_n$. Hence, for almost every $x \in X$, $P(n, T^\ell(x))$ is defined for all $n, \ell \in \mathbf{N}$. An easy consequence of the definition of $P(n,x)$ that we will use in the sequel is that

$$T(P(n,x)) = P(n-1, T(x)), \qquad n \geq 1. \tag{19}$$

If $x \in X_0$ then, since $P(n+1,x) \subset P(n,x)$ and $\operatorname{diam}(P(n,x)) \to 0$ when $n \to \infty$, we have that

$$\bigcap_n P(n,x) = \{x\}$$

and so the sequence $\{P(n,x)\}_n$ determines to the point x. Moreover, from (19) we have that $T^n(P(n,x)) = P(0, T^n(x))$, and it is not difficult to see that

$$\begin{aligned}
P(k,x) &= T\big|^{-1}_{P(0,x)} T\big|^{-1}_{P(0,T(x))} \cdots T\big|^{-1}_{P(0,T^{k-1}(x))} \Big(P(0,T^k(x))\Big) \\
&= T\big|^{-1}_{P(0,x)} T\big|^{-1}_{P(0,T(x))} \cdots T\big|^{-1}_{P(0,T^{k-2}(x))} \Big(T^{-1}\big(P(0,T^k(x))\big) \cap P(0,T^{k-1}(x))\Big) \\
&= T\big|^{-1}_{P(0,x)} T\big|^{-1}_{P(0,T(x))} \cdots T\big|^{-1}_{P(0,T^{k-3}(x))} \Big(T^{-2}\big(P(0,T^k(x))\big) \cap T^{-1}\big(P(0,T^{k-1}(x))\big) \cap P(0,T^{k-2}(x))\Big) \\
&= \ldots = \bigcap_{n=0}^{k} T^{-n}(P(0,T^n(x))).
\end{aligned}$$

Hence

$$\bigcap_{n=0}^{\infty} T^{-n}(P(0,T^n(x))) = \bigcap_{n=0}^{\infty} P(n,x) = \{x\}$$

and the sequence $\{P(0,T^n(x))\}_n$ also determines the point x.

We will also define the set X_0^+ as the union of X_0 with the set of points $x \in X$ verifying that there exists a sequence $\{P_n\}$, with $P_n \in \mathcal{P}_n$ and $P_{n+1} \subset P_n$, such that

$$\bigcap_{n=0}^{\infty} \operatorname{closure}(P_n) = \{x\}.$$

We remark that for points $x \in X_0^+ \setminus X_0$ the sequence $\{P_n\}$ is not uniquelly determinated by x. From now on, for each $x \in X_0^+ \setminus X_0$ we make an election of $\{P_n\}$ and we denote P_n by $P(n,x)$. Also by $P(0,T^n(x))$ we mean $T^n(P(n,x))$. We are extending in this way the definition of $P(n,x)$ and $P(0,T^n(x))$ given for points in X_0 in such a way that for points in $X_0^+ \setminus X_0$ we also have that $T^n(P(n,x)) = P(0,T^n(x))$.

Definition 4.1. *If $x \in X_0^+$, then we will code x as the sequence $\{i_0, i_1, \ldots\}$ and we will write $x = [\, i_0\ i_1\ \ldots\,]$ if and only if*

$$P(0,T^n(x)) = P_{i_n} \in \mathcal{P}_0, \qquad \textit{for all} \quad n = 0,1,2,\ldots$$

Remark 4.5. *If $x = [\, i_0\ i_1\ i_2 \ldots\,]$ then $T(x) = [\, i_1\ i_2\ i_3\ \ldots\,]$. Therefore T acts as the left shift on the space of all codes.*

4.2 Some properties of expanding maps

Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system. Following [30] we have

Proposition 4.1. *There exists an absolute constant $C > 0$ such that for all $x_0 \in X_0^+$ and for all natural number n we have that if $x, y \in P(n, x_0)$ then*

$$\frac{\mathbf{J}_s(x)}{\mathbf{J}_s(y)} \leq C\,, \qquad \text{for } s = 1, \dots, n\,. \tag{20}$$

Moreover, if $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then (20) holds for $s = n + 1$.

Proof. We will prove the lemma for the case $s = n+1$. If $x, y \in P(n.x_0)$ we have, from properties (B) and (C), that

$$\begin{aligned}
\frac{\mathbf{J}_{n+1}(x)}{\mathbf{J}_{n+1}(y)} &= \prod_{k=0}^{n} \frac{\mathbf{J}(T^k(x))}{\mathbf{J}(T^k(y))} \leq \prod_{k=0}^{n} (1 + C\, d(T^{k+1}(x), T^{k+1}(y))^\alpha) \\
&= (1 + C\, d(T^{n+1}(x), T^{n+1}(y))^\alpha) \prod_{k=0}^{n-1} (1 + C\, d(T^{k+1}(x), T^{k+1}(y))^\alpha) \\
&\leq (1 + C\, [\operatorname{diam} T(P(0, T^n(x_0)))]^\alpha) \prod_{k=0}^{n-1} (1 + C\, \beta^{-\alpha(n-(k+1))} d(T^n(x), T^n(y))^\alpha)
\end{aligned}$$

since $x, y \in P(n, x_0)$ implies that $T^{k+1}(x), T^{k+1}(y) \in P(n-(k+1), T^{k+1}(x_0))$ for $k = 0, \dots, n-1$. Therefore,

$$\begin{aligned}
\frac{\mathbf{J}_{n+1}(x)}{\mathbf{J}_{n+1}(y)} &\leq (1 + C\, [\operatorname{diam} T(P(0, T^n(x_0)))]^\alpha) \prod_{j=0}^{n-1} (1 + C\, \beta^{-\alpha j} [\operatorname{diam} P(0, T^n(x_0))]^\alpha) \\
&\leq (1 + C\, \overline{D}^\alpha) \exp\left[C\, D^\alpha \sum_{j=0}^{\infty} \beta^{-\alpha j}\right] \leq C\,,
\end{aligned}$$

where $D = \sup_{P\in\mathcal{P}_0} \operatorname{diam}(P)$ and $\overline{D} = \sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P))$. □

An easy consequence of the above bound, that we will often use, is the following one:

Proposition 4.2. *If P is an element of $\mathcal{P}_n$, i.e. if $P = P(n, x)$ for some $x \in X$, and P' is a measurable subset of P then*

$$\frac{1}{C} \frac{\lambda(T^j(P'))}{\lambda(T^j(P))} \leq \frac{\lambda(P')}{\lambda(P)} \leq C \frac{\lambda(T^j(P'))}{\lambda(T^j(P))}\,, \qquad \text{for } j = 1, \dots, n\,,$$

with C an absolute constant. Moreover, if $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then the above inequality is true for $j = n + 1$.

Proof. Using (17) we get

$$\frac{\inf_{y\in P} \mathbf{J_j}(y)}{\sup_{x\in P} \mathbf{J_j}(x)} \frac{\lambda(P')}{\lambda(P)} \leq \frac{\lambda(T^j(P'))}{\lambda(T^j(P))} = \frac{\int_{P'} \mathbf{J_j}\, d\lambda}{\int_P \mathbf{J_j}\, d\lambda} \leq \frac{\sup_{x\in P} \mathbf{J_j}(x)}{\inf_{y\in P} \mathbf{J_j}(y)} \frac{\lambda(P')}{\lambda(P)}$$

and the result follows from Proposition 4.1. □

Lemma 4.1. *If $A \in \mathcal{A}$ and $Q \in \mathcal{P}_m$ for some m, then*

$$\lambda(T^{-\ell}(A) \cap Q) = \int_A \sum_{y\in T^{-\ell}(x)\cap Q} \frac{1}{\mathbf{J}_\ell(y)}\, d\lambda(x)\,, \qquad \text{for } \ell = 1, 2, \dots\,.$$

Proof. We may assume that $T^{-\ell}(A) \cap Q \neq \emptyset$ and $A \subseteq P \in \mathcal{P}_m$. The general result follows from the fact that $\mathcal{P}_m$ is a partition of X. Then we have that $T^{-\ell}(A)\cap Q$ is a union of some elements $B_1, B_2, \dots$ such that $B_i \subset P_i \in \mathcal{P}_{\ell+m}$ and $T^\ell\big|_{B_j} : B_j \longrightarrow A$ is bijective for all j. Let us denote by S_j its inverse map, $S_j : A \longrightarrow B_j$. Then

$$\lambda(T^{-\ell}(A) \cap Q) = \sum_j \lambda(B_j) = \sum_j \lambda(S_j(A))\,.$$

But using (17) we deduce that

$$\lambda(S_j(A)) = \int_{S_j(A)} d\lambda = \int_{S_j(A)} \frac{\mathbf{J}_\ell(x)}{\mathbf{J}_\ell((S_j(T^\ell(x)))} \, d\lambda(x) = \int_{T^\ell(S_j(A))} \frac{1}{\mathbf{J}_\ell(S_j(x))} \, d\lambda(x) \, .$$

Therefore, since $T^\ell(S_j(A)) = A$, we have

$$\lambda(T^{-\ell}(A) \cap Q) = \sum_j \int_A \frac{1}{\mathbf{J}_\ell(S_j(x))} \, d\lambda(x) = \int_A \sum_j \frac{1}{\mathbf{J}_\ell(S_j(x))} \, d\lambda(x) \, .$$

If we denote, for each j, $y = S_j(x)$ we have that $y \in T^{-\ell}(x) \cap B_j$ and that y is unique. This observation completes the proof. □

Lemma 4.2. *If $x \in P_0 \in \mathcal{P}_0$ and $z \in Q \in \mathcal{P}_m$, then*

$$\sum_{y \in T^{-\ell}(x) \cap Q} \frac{1}{\mathbf{J}_\ell(y)} \leq \begin{cases} C\,\lambda(P(\ell, z)) & \text{if } \ell < m, \\ \\ C\,\lambda(Q) & \text{if } \ell \geq m. \end{cases}$$

with $C > 0$ a constant depending on P_0.

Proof. Using (19), (17) and Proposition 4.1 we deduce that

$$\lambda(P_0) = \lambda(P(0, x)) = \lambda(P(0, T^\ell(y)) = \lambda(T^\ell(P(\ell, y))) = \int_{P(\ell, y)} \mathbf{J}_\ell \, d\lambda \asymp \mathbf{J}_\ell(y) \lambda(P(\ell, y)) \, .$$

Therefore

$$\frac{1}{\mathbf{J}_\ell(y)} \asymp \frac{\lambda(P(\ell, y))}{\lambda(P_0)} \, . \tag{21}$$

If $\ell \geq m$ we have that $P(\ell, y) \subset Q$ for all $y \in T^{-\ell}(x) \cap Q$ and so

$$\sum_{y \in T^{-\ell}(x) \cap Q} \frac{1}{\mathbf{J}_\ell(y)} \asymp \frac{1}{\lambda(P_0)} \sum_{y \in T^{-\ell}(x) \cap Q} \lambda(P(\ell, y)) \leq \frac{C}{\lambda(P_0)} \, \lambda(Q)$$

On the other hand, if $\ell < m$ the map T^ℓ is injective in Q and therefore there is at most one point $y \in T^{-\ell}(x) \cap Q$. Since $P(\ell, y) \supset P(m, y) = Q$ we also have that $P(\ell, z) = P(\ell, y)$ for any $z \in Q$. Therefore, in this case, the result follows from (21). □

Remark 4.6. Under the same hypotheses for x and Q, and if

$$C_0 := \inf_{P \in \mathcal{P}_0} \lambda(T(P)) > 0 \qquad \text{and} \qquad D_0 := \sup_{P \in \mathcal{P}_0} \operatorname{diam}(T(P)) < \infty,$$

then

$$\sum_{y \in T^{-\ell}(x) \cap Q} \frac{1}{\mathbf{J}_\ell(y)} \leq \begin{cases} C\,\lambda(P(\ell - 1, z)) & \text{if } \ell < m + 1, \\ \\ C\,\lambda(Q) & \text{if } \ell \geq m + 1. \end{cases}$$

with C a constant depending on C_0 and D_0.

Proof. Notice that from Proposition 4.1 we get that

$$\lambda(T(P(0, T^{\ell-1}(y)))) = \lambda(T^\ell(P(\ell - 1, y))) = \int_{P(\ell-1, y)} \mathbf{J}_\ell \, d\lambda \asymp \mathbf{J}_\ell(y) \lambda(P(\ell - 1, y)) \, .$$

Therefore,

$$\frac{1}{\mathbf{J}_\ell(y)} \asymp \frac{\lambda(P(\ell - 1, y))}{\lambda(T(P(0, T^{\ell-1}(y))))} \leq \frac{1}{C_0} \, \lambda(P(\ell - 1, y)) \, .$$

The rest of the proof is similar to the proof of Lemma 4.2. □

Proposition 4.3. *Let μ be the ACIPM measure associated to the expanding system. Let $A \in \mathcal{A}$ and $Q \in \mathcal{P}_m$ with $A, Q \subset P_0 \in \mathcal{P}_0$. Then, we have that*

$$\mu(T^{-\ell}(A) \cap Q) \leq \begin{cases} C\,\mu(A)\mu(P(\ell, z)) & \text{if } \ell < m, \\ \\ C\,\mu(A)\mu(Q) & \text{if } \ell \geq m. \end{cases}$$

where z is any point of Q and $C > 0$ is a constant depending on P_0.

Proof. By Theorem E we know that

$$\mu(V) \asymp \lambda(V) \text{ for all measurable set } V \subset P_0 \tag{22}$$

and the result is a consequence of lemmas 4.1 and 4.2. □

Remark 4.7. If $\inf_{P\in\mathcal{P}_0} \lambda(T(P)) > 0$ and $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then λ and μ are comparable in the whole X and it is not necessary in the statement of the Proposition 4.3 that $A, Q \subset P_0$.

Recall now the definitions of lower and upper $\mathcal{P}$-dimensions, see Definition 1.1. Since the sequence $\{P(n, x_0)\}_{n\in\mathbf{N}}$ is defined for all $x_0 \in X_0^+$, we have that $\overline{\delta}_\lambda(x_0)$ and $\underline{\delta}_\lambda(x_0)$ are also defined for $x_0 \in X_0^+$,

Lemma 4.3. *Let $x_0 \in X_0^+$ such that $\underline{\delta}_\lambda(x_0) > 0$. Given $0 < \varepsilon < \underline{\delta}_\lambda(x_0)$ there exists $N \in \mathbf{N}$ such that for all $n \geq N$*

$$\lambda(P(n, x_0)) \leq \beta^{-n(\underline{\delta}_\lambda(x_0)-\varepsilon)}$$

with $\beta > 1$ the constant in the property (C) *of expanding maps.*

Proof. By definition of $\underline{\delta}_\lambda(x_0)$ we have that for n large enough

$$\lambda(P(n, x_0)) \leq (\operatorname{diam}(P(n, x_0)))^{\underline{\delta}_\lambda(x_0)-\varepsilon/2}$$

Now, if $x_0 \in X_0$, from the property (C) of expanding maps we get that

$$C_2\beta^n \operatorname{diam}(P(n, x_0)) \leq \operatorname{diam}(P(0, T^n(x_0))) \leq D = \sup_{P\in\mathcal{P}_0} \operatorname{diam}(P) < \infty.$$

The result follows for $x_0 \in X_0$ from these two inequalities. If $x_0 \in X_0^+ \setminus X_0$, then $P(n, x_0) = P(n, x)$ with $x \in P(n, x_0) \cap X_0$ and from this fact we conclude that also $C_2\beta^n \operatorname{diam}(P(n, x_0)) \leq \sup_{P\in\mathcal{P}_0} \operatorname{diam}(P)$ for these points. □

Another quantity that we will need is the following:

Definition 4.2. *The rate of decay of the measure λ at $x \in X_0$ with respect to the partition $\mathcal{P}_0$ is defined as*

$$\overline{\tau}_\lambda(x) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{\lambda(P(n, x))}{\lambda(P(n+1, x))}\,.$$

Notice that we can extend the definition of $\overline{\tau}_\lambda(x_0)$ to all $x_0 \in X_0^+$.

Lemma 4.4. *If the entropy $H_\mu(\mathcal{P}_0)$ of the partition $\mathcal{P}_0$ with respect to the ACIPM measure associated to the expanding system is finite, then the set of points $x_0 \in X_0^+$ verifying*

$$\overline{\tau}_\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \frac{\lambda(P(n, x_0))}{\lambda(P(n+1, x_0))} = 0 \tag{23}$$

has full λ-measure. Besides (23) *holds if $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$ and $x_0 \in X_0^+$ verifies*

$$\inf_j \lambda(P(0, T^j(x_0))) > 0\,.$$

In particular, if the partition $\mathcal{P}_0$ is finite, then (23) *holds for all $x_0 \in X_0^+$.*

Proof. . From part (iii) of Theorem E we know that the measures μ and λ are comparable in each element of the partition $\mathcal{P}_0$ and as a consequence the zero measure sets are the same for μ and λ. Hence from Theorem D we have that for λ-almost all $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda(P(n,x))} = \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu(P(n,x))} = h_\mu,$$

and therefore (23) holds.

Now let us prove that if $\inf_j \lambda(P(0,T^j(x_0))) > 0$ and $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then (23) holds for all $x_0 \in X_0^+$. By Proposition 4.2 and (19) we have that for all $x_0 \in X_0$

$$\frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} \le C\frac{\lambda(T(P(0,T^n(x_0))))}{\lambda(P(0,T^{n+1}(x_0)))} \le C\frac{\lambda(X)}{\inf_j \lambda(P(0,T^j(x_0)))} < C'$$

with $C' > 1$ a constant. This implies (23) for all $x_0 \in X_0$. If $x_0 \in X_0^+ \setminus X_0$, then $P(n+1,x_0) = P(n+1,x)$ for $x \in P(n+1,x_0) \cap X_0$ and, since $P(n+1,x_0) \subset P(n,x_0)$, we also have that $x \in P(n,x_0) \cap X_0$ and therefore also $P(n,x) = P(n,x_0)$. The result for these points follows now from the last chain of inequalities. $\square$

Lemma 4.5. *Let $x_0 \in X_0^+$ be a point such that $\overline{\delta}_\lambda(x_0) < \infty$ and $\overline{\tau}_\lambda(x_0) < \infty$. Given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that for all $n \ge N$*

$$\lambda(P(n,x_0)) \ge (\operatorname{diam}(P(n-1,x_0)))^{\overline{\delta}_\lambda(x_0)+\overline{\tau}_\lambda(x_0)/\log\beta+\varepsilon}$$

where $\beta > 1$ is the constant in the property (C) *of expanding maps.*

Proof. By Definition 4.2 we have that for n large enough

$$\frac{1}{n-1} \log \frac{\lambda(P(n-1,x_0))}{\lambda(P(n,x_0))} < \overline{\tau}_\lambda(x_0) + \frac{1}{3}\varepsilon \log\beta\,.$$

Hence, for n large enough,

$$\lambda(P(n,x_0)) \ge \beta^{-(n-1)\varepsilon/3} e^{-(n-1)\overline{\tau}_\lambda(x_0)} \lambda(P(n-1,x_0))\,. \tag{24}$$

But from the property (C) of expanding maps, if $x_0 \in X_0$ we have that,

$$C_2\beta^{n-1}\operatorname{diam}(P(n-1,x_0)) \le \operatorname{diam}(P(0,T^{n-1}x_0)) \le D$$

with $D = \sup_{P\in\mathcal{P}_0} \operatorname{diam}(P)$. If $x_0 \in X_0^+\setminus X_0$, we obtain the same conclusion since $P(n-1,x_0) = P(n-1,x)$ for $x \in P(n-1,x_0) \cap X_0$. Therefore, in any case, we get that

$$\beta^{-(n-1)\varepsilon/3} e^{-(n-1)\overline{\tau}_\lambda(x_0)} \ge C(\operatorname{diam}(P(n-1,x_0)))^{\varepsilon/3+\overline{\tau}_\lambda(x_0)/\log\beta} \tag{25}$$

with $C > 0$.

Finally from the definition of $\overline{\delta}_\lambda(x_0)$ we have that for n large

$$\lambda(P(n-1,x_0)) \ge \operatorname{diam}(P(n-1,x_0))^{\overline{\delta}_\lambda(x_0)+\varepsilon/3}\,. \tag{26}$$

The result follows from (24), (25), and (26). $\square$

Using lemma 3.1 we can define an important subset of X_0 which also has full λ-measure. We will refer to this set in the rest of the paper. The following lemma summarizes its properties.

Lemma 4.6. *Let $(X, \mathcal{A}, \lambda, T)$ be an expanding system such that the entropy $H_\mu(\mathcal{P}_0)$ of the partition $\mathcal{P}_0$, with respect to the unique T-invariant probability measure which is absolutely continuous with respect to λ, is finite. Let X_1 denote the subset of X_0*

$$X_1 = X_0 \setminus [\cup_{m=1}^\infty \cap_N E_N^{1/m}]\,.$$

with $\{E_N^{1/m}\}$ the sets given by Lemma 3.1 for $\varepsilon = 1/m$. Then $\lambda(X_1) = \lambda(X)$ and moreover, if $x_0 \in X_1$ then:

(i) *$P(n, T^\ell(x_0))$ is well defined for all $n, \ell \in \mathbf{N}$.*

(ii) *For all positive integer m there exists $N \in \mathbf{N}$ such that for all $n \geq N$*

$$\frac{1}{M}\, e^{-n(h_\mu+1/m)} < \lambda(P(n,x_0)) < M\, e^{-n(h_\mu-1/m)} \tag{27}$$

with $M > 0$ depending on $P(0, x_0)$.

(iii)

$$\overline{\tau}_\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} = 0\,.$$

(iv) $\overline{\delta}_\lambda(x_0) \leq h_\mu/\log\beta < \infty$*, with β the constant given by property* (C) *of expanding maps.*

Proof. In the proof of Lemma 3.1 we saw that $\mu(\cap_N E_N^{1/m}) = 0$. Then, by Theorem E, we obtain that $\lambda(\cap_N E_N^{1/m}) = 0$, and therefore $\lambda[\cup_{m=1}^{\infty} \cap_N E_N^{1/m}] = 0$. Hence we have that $\lambda(X_1) = \lambda(X_0)$, but when we defined X_0 (see (18)) we showed that $\lambda(X_0) = \lambda(X)$.

The property (i) is satisfied for all points in X_0 and therefore also in X_1. If the point $x_0 \in X_1$, then, for all positive integer m, x_0 does not belong to $\cap_N E_N^{1/m}$, and so from Lemma 3.1 we have that for all n large enough

$$e^{-n(h_\mu+1/m)} < \mu(P(n,x_0)) < e^{-n(h_\mu-1/m)}\,.$$

From part (iii) of Theorem E, we conclude that (27) holds.

Moreover, from (27) we also get that

$$\frac{1}{n} \log \frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} < \frac{2\log M + h_\mu + 1/m}{n} + \frac{2}{m} \longrightarrow \frac{2}{m} \qquad \text{as} \qquad n\to\infty$$

Therefore by taking $m \to \infty$, we get the property (iii).

By property (C) of expanding maps we also have that

$$\operatorname{diam}(P(n,x_0)) \leq C\,\beta^{-n}\,,$$

and therefore, using again (27), we get that

$$\frac{\log\lambda(P(n,x_0))}{\log\operatorname{diam}(P(n,x_0))} \leq \frac{n(h_\mu+1/m)+\log M}{n\log\beta - \log C} \longrightarrow \frac{h_\mu+1/m}{\log\beta}$$

as $n \to \infty$, and so by letting $m \to \infty$ we obtain that $\overline{\delta}_\lambda(x_0) \leq h_\mu/\log\beta < \infty$. □

5 Measure results

We want to study the size of the set

$$\mathcal{W}(x_0,\{r_n\}) = \{x \in X:\ d(T^n(x),x_0) < r_n \text{ for infinitely many } n\}$$

where $\{r_n\}$ is a given sequence of positive numbers and x_0 is an arbitrary point in X. Observe that if the sequence $\{r_n\}$ is constant this set is T-invariant, but, in general, this is not the case.

We are also interested in the size of another set that we will see that is closely related with $\mathcal{W}(x_0,\{r_n\})$. This set is

$$\widetilde{\mathcal{W}}(x_0,\{t_n\}) = \{x \in X:\ T^k(x) \in P(t_k,x_0) \text{ for infinitely many } k\}$$

with $\{t_k\}$ an increasing sequence of positive integers and $x_0 \in X_0^+$.

Notice also that if $x \in \widetilde{\mathcal{W}}(x_0,\{t_n\})$ then $P(m,x)$ is well defined for infinitely many m. and so it is well defined for all m. Therefore $\widetilde{\mathcal{W}}(x_0,\{t_n\}) \subset X_0$.

Let us denote

$$A_k = T^{-k}(B(x_0,r_k)) \qquad \text{and} \qquad \widetilde{A}_k = T^{-k}(P(t_k,x_0))$$

With these notations, we have that

$$\mathcal{W}(x_0, \{r_n\}) = \{x \in X : \ x \in A_n \text{ for infinitely many } n\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \,,$$

$$\widetilde{\mathcal{W}}(x_0, \{t_n\}) = \{x \in X : \ x \in \widetilde{A}_n \text{ for infinitely many } n\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \widetilde{A}_n \,.$$

The following result on the size of these sets is a consequence of the direct part of Borel-Cantelli lemma and Theorem E.

Proposition 5.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system.*

i) *Let $x_0 \in \cup_{P \in \mathcal{P}_0} P$ and let $\{r_n\}$ be a sequence of positive numbers.*

$$\textit{If} \qquad \sum_{n=1}^{\infty} \lambda(B(x_0, r_n)) < \infty \qquad \textit{then} \qquad \lambda(\mathcal{W}(x_0, \{r_n\})) = 0.$$

ii) *Let $x_0 \in X_0^+$ and let $\{t_n\}$ be a non decreasing sequence of positive integers.*

$$\textit{If} \qquad \sum_{n=1}^{\infty} \lambda(P(t_n, x_0)) < \infty, \qquad \textit{then} \qquad \lambda(\widetilde{\mathcal{W}}(x_0\{t_n\})) = 0.$$

Proof. i) First of all, let us observe that $\lim_{n\to\infty} r_n = 0$. Let μ be the ACIPM associated to the system. We have that $\mu(B(x_0, r_k)) = \mu(A_k)$ for all $k \in \mathbf{N}$ and therefore

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(B(x_0, r_n)) < \infty \,,$$

since for r_n small enough, $B(x_0, r_n) \subset P(0, x_0)$ and λ and μ are comparable in that set by Theorem E. From Borel-Cantelli lemma it follows that $\mu(\mathcal{W}(x_0, \{r_n\})) = 0$ and using the Remark 4.2, we conclude that $\lambda(\mathcal{W}(x_0, \{r_n\})) = 0$. The same argument works for part ii). $\square$

Corollary 5.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system. Let $x_0 \in \cup_{P \in \mathcal{P}_0} P$ and let $\{r_n\}$ be a sequence of positive numbers. If $\sum_{n=1}^{\infty} \lambda(B(x_0, r_n)) < \infty$ then*

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \geq 1 \,, \qquad \textit{for } \lambda\textit{-almost every } x \in X \,.$$

Corollary 5.2. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system. Let $x_0 \in \cup_{P \in \mathcal{P}_0} P$ such that*

$$0 < \underline{\Delta}_\lambda(x_0) := \liminf_{r\to 0} \frac{\log \lambda(B(x_0, r))}{\log r} < \infty \,,$$

and let $\{r_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} r_n^{\underline{\Delta}_\lambda(x_0)-\varepsilon} < \infty$ for some $0 < \varepsilon < \underline{\Delta}_\lambda(x_0)$. Then

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = \infty \,, \qquad \textit{for } \lambda\textit{-almost every } x \in X \,.$$

If there exists a constant $\Delta(x_0)$ such that $\lambda(B(x_0, r)) \leq C r^{\Delta(x_0)}$ for all r small enough, then the conclusion holds when $\sum_{n=1}^{\infty} r_n^{\Delta(x_0)} < \infty$.

Proof. By definition of $\underline{\Delta}_\lambda(x_0)$, we have that for any r small enough

$$\lambda(B(x_0, r)) \leq r^{\underline{\Delta}_\lambda(x_0)-\varepsilon} \,.$$

Now, for any $m \in \mathbf{N}$, since $\lim_{n\to\infty} r_n = 0$, we have that for n big enough, (depending on m),

$$\lambda(B(x_0, m r_n)) \leq (m r_n)^{\underline{\Delta}_\lambda(x_0)-\varepsilon} \,.$$

Therefore, for all $m \in \mathbf{N}$,

$$\sum_n \lambda(B(x_0, m r_n)) \leq C_m \, m^{\underline{\Delta}_\lambda(x_0)-\varepsilon} \sum_n r_n^{\underline{\Delta}_\lambda(x_0)-\varepsilon} < \infty \,.$$

From Corollary 5.1 we get that, for all $m \in \mathbf{N}$,

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \geq m\,, \qquad \text{for } \lambda\text{-almost every } x \in X\,.$$

The result follows now from the fact that

$$\left\{x \in X:\ \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = \infty\right\} = \bigcap_{m=1}^{\infty} \left\{x \in X:\ \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \geq m\right\}.$$

□

Remark 5.1. If the measures λ and μ are comparable in X, then part i) of Proposition 5.1 and its corollaries hold for all $x_0 \in X$. In particular, this happens if

$$\inf_{P\in\mathcal{P}_0} \lambda(T(P)) > 0 \qquad \text{and} \qquad \sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty\,,$$

see Remark 4.3. Also, part i) of Proposition 5.1 and its corollaries hold for those x_0 such that the set of elements $P \in \mathcal{P}_0$ such that x_0 belongs to ∂P is finite. For example, if $X \subseteq \mathbf{R}$ or if the partition $\mathcal{P}_0$ is finite, then all $x_0 \in X$ satisfy the above condition.

Theorem 5.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system. Let x_0 be a point of X_0^+ such that $\underline{\delta}_\lambda(x_0) > 0$ and let $\{t_n\}$ be a non decreasing sequence of positive integers numbers.*

$$\textit{If} \qquad \sum_{n=1}^{\infty} \lambda(P(t_n, x_0)) = \infty, \qquad \textit{then} \qquad \lambda(\widetilde{\mathcal{W}}(x_0, \{t_n\})) = \lambda(X).$$

Moreover, if the partition $\mathcal{P}_0$ is finite or if the system has the Bernoulli property, (i.e. if $T(P) = X$ (mod 0) for all $P \in \mathcal{P}_0$), then we have the following quantitative version:

$$\lim_{n\to\infty} \frac{\#\{i \leq n:\ T^i(x) \in P(t_i, x_0)\}}{\sum_{j=1}^{n} \mu(P(t_j, x_0))} = 1\,, \qquad \textit{for } \lambda\textit{-almost every } x \in X.$$

In the proof of Theorem 5.1 we will use the following classical result.

Lemma (Payley-Zygmund Inequality). *Let $(X, \mathcal{A}, \mu)$ be a probability space and let $Z : X \longrightarrow \mathbf{R}$ be a positive random variable. Then, for $0 < \tau < 1$,*

$$\mu[Z > \tau\, E(Z)] \geq (1-\tau)^2 \frac{E(Z)^2}{E(Z^2)}\,,$$

where $E(\cdot)$ denotes expectation value.

Proof of Theorem 5.1. Let μ be the ACIPM associated to the system. For $j \geq k$, we have that

$$\mu(\widetilde{A}_k \cap \widetilde{A}_j) = \mu(T^{-k}[P(t_k, x_0)) \cap T^{-(j-k)}(P(t_j, x_0))]) = \mu(P(t_k, x_0)) \cap T^{-(j-k)}(P(t_j, x_0)))\,,$$

and by using Proposition 4.3 with $\ell = j - k$, $n = t_j$ and $m = t_k$, and using again that T preserves the measure μ, we conclude that

$$\mu(\widetilde{A}_k \cap \widetilde{A}_j) \leq \begin{cases} C\mu(\widetilde{A}_j)\mu(P(j-k, x_0)) & \text{if } j-k < t_k, \\ \\ C\mu(\widetilde{A}_j)\mu(\widetilde{A}_k) & \text{if } j-k \geq t_k. \end{cases} \tag{28}$$

with $C > 0$ depending on $P(0, x_0)$. Let us denote by Z_n and Z the counting functions

$$Z_n = \sum_{k=1}^{n} \chi_{\widetilde{A}_k} \qquad \text{and} \qquad Z = \sum_{k=1}^{\infty} \chi_{\widetilde{A}_k}\,,$$

where $\chi_{\widetilde{A}_k}$ is the characteristic function of $\widetilde{A}_k$. Observe that $\widetilde{\mathcal{W}}(x_0, \{r_n\}) = \{x \in X_0:\ Z(x) = \infty\}$.

If we compute the expectation value of Z_n^2 (with respect to μ), we obtain

$$E(Z_n^2) = E\Big[\sum_{k=1}^{n} \chi_{\widetilde{A}_k} + \sum_{\substack{k,j=1\\k\neq j}}^{n} \chi_{\widetilde{A}_k\cap\widetilde{A}_j}\Big] = \sum_{k=1}^{n}\mu(\widetilde{A}_k) + 2\sum_{\substack{k,j=1\\k<j}}^{n}\mu(\widetilde{A}_k\cap\widetilde{A}_j)$$

and using (28) we get

$$E(Z_n^2) \leq E(Z_n) + 2C\sum_{\substack{k,j=1\\k<j}}^{n}\mu(\widetilde{A}_j)\,\mu(P(j-k,x_0) + 2C\sum_{\substack{k,j=1\\k<j}}^{n}\mu(\widetilde{A}_j)\,\mu(\widetilde{A}_k)\,.$$

But $\mu(\widetilde{A}_j) \leq \mu(\widetilde{A}_k)$ for all $j > k$ because $\{t_n\}$ is non decreasing. Therefore

$$E(Z_n^2) \leq E(Z_n) + 2C\sum_{k=1}^{n}\mu(\widetilde{A}_k)\sum_{j=k+1}^{n}\mu(P(j-k,x_0)) + C\,E(Z_n)^2\,. \tag{29}$$

Since by Theorem E the measures λ and μ are comparable in $P(0,x_0)$ we get from Lemma 4.3 that

$$\sum_{s=1}^{\infty}\mu(P(s,x_0)) \leq C\sum_{s=1}^{\infty}\beta^{-s(\underline{\delta}_\lambda(x_0)-\varepsilon)} \leq C' \tag{30}$$

with C' a positive constant. From (29), and (30) we obtain that

$$E(Z_n^2) \leq \Big(1 + 2CC'\Big)\,E(Z_n) + C\,E(Z_n)^2\,. \tag{31}$$

By applying Paley-Zygmund Lemma we obtain from (31) that

$$\begin{aligned}\mu[\{x\in X:\ Z(x) > \tau\,E(Z_n)\}] &\geq \mu[\{x\in X:\ Z_n(x) > \tau\,E(Z_n)\}]\\ &\geq (1-\tau)^2\frac{E(Z_n)}{1+2CC'+C\,E(Z_n)}\,.\end{aligned} \tag{32}$$

Using again that λ and μ are comparable in $P(0,x_0)$, we get that

$$E(Z_n) = \sum_{k=1}^{n}\mu(\widetilde{A}_k) = \sum_{k=1}^{n}\mu(P(t_k,x_0)) \geq C\sum_{k=1}^{n}\lambda(P(t_k,x_0))$$

and from the hypothesis of the theorem, we obtain that $E(Z_n) \to \infty$ as $n \to \infty$. Hence, we have from (32) that

$$\mu[\{x\in X:\ Z(x) = \infty\}] \geq \frac{1}{C}(1-\tau)^2\,, \qquad \text{for } 0<\tau<1\,.$$

and we conclude that $\widetilde{\mathcal{W}}(x_0,\{t_n\})$ has positive μ-measure. If we denote, for each $n\in\mathbf{N}$

$$\widetilde{\mathcal{W}}_n(x_0,\{t_n\}) = \{x\in X:\ T^{k-n}(x)\in P(t_k,x_0) \text{ for infinitely many } k \text{ with } k\geq n\}\,,$$

it is easy to see that

$$\widetilde{\mathcal{W}}(x_0,\{t_n\}) = T^{-n}(\widetilde{\mathcal{W}}_n(x_0),\{t_n\}) \qquad \text{for each } n\in\mathbf{N}$$

and since T is exact with respect to μ (see Theorem E) it follows that $\widetilde{\mathcal{W}}(x_0,\{t_n\})$ has full μ-measure. Therefore from Remark 4.2 we conclude that $\widetilde{\mathcal{W}}(x_0,\{t_n\})$ has full λ-measure.

Finally, if the system has the Bernoulli property then the correlation coefficients of the sets $\{P(n,x_0)\}_{n\in\mathbf{N}}$ have exponential decay, see [44]. Concretely, she proves that

$$|\mu(T^{-\ell}(P(n,x_0))\cap P(m,x_0)) - \mu(P(n,x_0))\mu(P(n,x_0))| \leq C\,\mu(P(n,x_0))\,e^{-\alpha\ell} \tag{33}$$

for some absolute positive constants C and α and for all $m,n,\ell\in\mathbf{N}$. The same argument used in the proof of Theorem 1 in [19], gives the quantitative version.

If the partition $\mathcal{P}_0$ is finite, then the dynamical system $(X,\mathcal{A},\mu,T)$ is isomorphic via coding to a (one-sided) subshift of finite type. The stochastic matrix M of this subshift is defined in the following way: $p_{i,j} = \mu(T^{-1}(P_j)\cap P_i)/\mu(P_i)$ where $P_i, P_j\in\mathcal{P}_0$. Property (A.7) implies that M verifies that M^{n_0} has all its entries positive, see for example [28], p.158 or Lemma 12.2 in [30]. This implies that the shift σ is mixing, see for example Proposition 12.3 in [30], and moreover (33) follows from the Perron-Frobenius theorem (see, for example [28] or [30]; see also [10]). $\square$

We state now the following corollary of this proof.

Corollary 5.3. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let $\{t_n\}$ be a non decreasing sequence of positive integers.*

Then for λ-almost all point $x_0 \in X_0^+$, more concretely if $x_0 \in X_1$ (see definition in Lemma 4.6), we have that

$$\text{If} \qquad \sum_{n=1}^{\infty} \lambda(P(t_n, x_0)) = \infty \qquad \text{then} \qquad \lambda(\widetilde{\mathcal{W}}(x_0, \{t_n\})) = \lambda(X).$$

Proof. If $x_0 \in X_1$ we have from Lemma 4.6 that for all $m \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that

$$e^{-n(h_\mu+1/m)} \leq \mu(P(n, x_0)) \leq e^{-n(h_\mu-1/m)}$$

for all $n \geq N$. Therefore we can substitute the inequality (30) by

$$\sum_{s=1}^{\infty} \mu(P(s, x_0)) \leq C \sum_{s=1}^{\infty} e^{-s(h_\mu-1/m)} \leq C' < \infty\,,$$

since $h_\mu > 0$ (see Remark 4.4) and we can take m large enough so that $0 < 1/m < h_\mu$. Hence we do not need now the hypothesis $\underline{\delta}_\lambda(x_0) > 0$. □

Theorem 5.2. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system. Let x_0 be a point of X_0^+ such that*

$$\overline{\tau}_\lambda(x_0) < \infty \qquad \text{and} \qquad 0 < \underline{\delta}_\lambda(x_0) \leq \overline{\delta}_\lambda(x_0) < \infty\,,$$

and let $\{r_n\}$ be a non increasing sequence of positive numbers.

$$\text{If} \qquad \sum_{n=1}^{\infty} r_n^{\overline{\delta}_\lambda(x_0)+\overline{\tau}_\lambda(x_0)/\log\beta+\varepsilon} = \infty \quad \text{for some } \varepsilon > 0, \qquad \text{then} \qquad \lambda(\mathcal{W}(x_0, \{r_n\})) = \lambda(X)\,.$$

Moreover, if the partition $\mathcal{P}_0$ is finite or if the system has the Bernoulli property, (i.e. if $T(P) = X (mod\ 0)$ for all $P \in \mathcal{P}_0$), then we have the following quantitative version:

$$\liminf_{n\to\infty} \frac{\#\{i \leq n : \ d(T^i(x), x_0) \leq r_i\}}{\sum_{j=1}^{n} r_j^{\overline{\delta}_\lambda(x_0)+\overline{\tau}_\lambda(x_0)/\log\beta+\varepsilon}} \geq C\,, \qquad \text{for } \lambda\text{-almost every } x \in X,$$

with C a positive constant depending on x_0 and on the comparability constants between λ and μ at $P(0, x_0)$.

Remark 5.2. If the correlation coefficients of the balls $\{B(x_0, r_n)\}$ had exponential decay, i.e. if they verify the relations

$$|\mu(T^{-\ell}(B(x_0, r_n)) \cap B(x_0, r_m)) - B(x_0, r_n)B(x_0, r_m)| \leq C\,\mu(B(x_0, r_n))\,e^{-\alpha\ell}$$

for some absolute positive constants C and α and for all $n, \ell \in \mathbf{N}$, then using the same arguments that in Theorem 1 in [19] we would have

$$\liminf_{n\to\infty} \frac{\#\{i \leq n : \ d(T^i(x), x_0) \leq r_i\}}{\sum_{j=1}^{n} \mu(B(x_0, r_j)} = 1\,, \qquad \text{for } \lambda\text{-almost every } x \in X,$$

Remark 5.3. We recall that by Lemma 4.4 we know that $\overline{\tau}_\lambda(x_0) = 0$ for λ-almost all $x_0 \in X$. We have also that $\overline{\tau}_\lambda(x_0) = 0$ if $\inf_j \lambda(P(0, T^j(x_0))) > 0$ and $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$. In particular, if the partition $\mathcal{P}_0$ is finite and $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then $\overline{\tau}_\lambda(x_0) = 0$ for all $x_0 \in X_0^+$.

Corollary 5.4. *Under the same hypothesis than Theorem 5.2, if*

$$\sum_{n=1}^{\infty} r_n^{\overline{\delta}_\lambda(x_0)+\overline{\tau}_\lambda(x_0)/\log\beta+\varepsilon} = \infty, \qquad \text{for some} \quad \varepsilon > 0\,, \tag{34}$$

then,

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0\,, \qquad \text{for } \lambda\text{-almost all } x \in X.$$

Proof. From Theorem 5.2 it follows easily that if the radii r_n verify (34) then

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq 1, \qquad \text{for } \lambda\text{-almost all } x \in X.$$

But notice that for any $m \in \mathbf{N}$ the radii r_n/m also verify (34) and therefore we get that for any $m \in \mathbf{N}$

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq \frac{1}{m}, \qquad \text{for } \lambda\text{-almost all } x \in X.$$

The result follows now from the fact that

$$\left\{x \in X: \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0\right\} = \bigcap_{m=1}^{\infty} \left\{x \in X: \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq \frac{1}{m}\right\}.$$

□

Proof of Theorem 5.2. Let μ be the ACIPM associated to the system. Given $x_0 \in X_0^+$ and the sequence r_k we define t_k as the smallest integer so that

$$P(t_k, x_0) \subset B(x_0, r_k). \tag{35}$$

Hence, $\widetilde{\mathcal{W}}(x_0, \{t_n\}) \subset \mathcal{W}(x_0, \{r_n\})$.

Moreover, since $\overline{\delta}_\lambda(x_0) < \infty$ and $\overline{\tau}_\lambda(x_0) < \infty$, from Lemma 4.5, we get that

$$\sum_{k=1}^{n} \lambda(P(t_k, x_0)) \geq \sum_{k=1}^{n} (\mathrm{diam}(P(t_k - 1, x_0)))^{\overline{\delta}_\lambda(x_0) + \overline{\tau}_\lambda(x_0)/\log\beta + \varepsilon} \tag{36}$$

But from the definition of t_k we have that

$$P(t_k - 1, x_0) \not\subset B(x_0, r_k)$$

and since $x_0 \in P(t_k - 1, x_0)$ we can conclude that

$$\mathrm{diam}(P(t_k - 1, x_0)) \geq r_k.$$

Therefore we get that

$$\sum_{k=1}^{\infty} \lambda(P(t_k, x_0)) \geq \sum_{k=1}^{\infty} r_k^{\overline{\delta}_\lambda(x_0) + \overline{\tau}_\lambda(x_0)/\log\beta + \varepsilon} = \infty$$

and from Theorem 5.1 we conclude that $\lambda(\mathcal{W}(x_0, \{r_n\})) = \lambda(X)$.

Now from (35), (36) and the fact that λ and μ are comparable on $P(0, x_0)$ we have that

$$\frac{\#\{i \leq n: d(T^i(x), x_0) \leq r_i\}}{\sum_{j=1}^{n} r_j^{\overline{\delta}_\lambda(x_0) + \overline{\tau}_\lambda(x_0)/\log\beta + \varepsilon}} \geq C \, \frac{\#\{i \leq n: T^i(x) \in P(t_i, x_0)\}}{\sum_{j=1}^{n} \mu(P(t_j, x_0))}.$$

Hence, the quantitative version follows from Theorem 5.1. □

We have also the following corollary of the proof of Theorem 5.2.

Corollary 5.5. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let $\{r_n\}$ be a non increasing sequence of positive numbers. Then for λ-almost all point $x_0 \in X$, more concretely if $x_0 \in X_1$ (see definition in Lemma 4.6), we have that*

$$\text{if} \quad \sum_{n=1}^{\infty} r_n^{\overline{\delta}_\lambda(x_0) + \varepsilon} = \infty \quad \text{for some } \varepsilon > 0, \quad \text{then} \quad \lambda(\mathcal{W}(x_0, \{r_n\})) = \lambda(X).$$

In particular, we conclude that, for λ-almost all point $x_0 \in X$,

$$\liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0, \qquad \text{for } \lambda\text{-almost all } x \in X.$$

Proof. As in the proof of Theorem 5.2 we define t_k by (35) and as a consequence $\widetilde{\mathcal{W}}(x_0, \{t_n\}) \subset \mathcal{W}(x_0, \{r_n\})$. Since $x_0 \in X_1$ we have, from Lemma 4.6, that $\overline{\tau}_\lambda(x_0) = 0$ and $\overline{\delta}_\lambda(x_0) < \infty$ and therefore we get the inequality (36) with $\overline{\tau}_\lambda(x_0) = 0$.

From Lemma 4.6 we also have that for all $m \in \mathbf{N}$ there exists $N \in \mathbf{N}$ such that

$$e^{-n(h_\mu + 1/m)} \leq \mu(P(n, x_0)) \leq e^{-n(h_\mu - 1/m)}$$

for all $n \geq N$. The same argument given in Corollary 5.3 allows us to avoid the condition $\underline{\delta}_\lambda(x_0) > 0$. The fisrt part of the corollary follows from these facts as in Theorem 5.2. The last assertion follows from Corollary 5.4. $\square$

Corollary 5.6. *Under the same hypotheses than Corollary* 5.5 *we have that if* $\underline{\Delta}_\lambda(x_0) = \overline{\delta}_\lambda(x_0) := D(x_0) := D$ *and*

$$\sum_{k=1}^{\infty} \lambda(B(x_0, r_k))^{1+\varepsilon} = \infty \quad \text{for some } \varepsilon > 0, \quad \text{then} \qquad \lambda(\mathcal{W}(x_0)) = \lambda(X)\,.$$

Proof. From the definition of $\underline{\Delta}_\lambda(x_0)$ (see Corollary 5.2), the condition $\sum_{k=1}^{\infty} \lambda(B(x_0, r_k))^{1+\varepsilon} = \infty$ implies that $\sum_{k=1}^{\infty} r_k^{(1+\varepsilon)(D-\varepsilon')} = \infty$. But if ε' is small enough we have that $(1+\varepsilon)(D-\varepsilon') = D + \varepsilon''$ with $\varepsilon'' > 0$. Since $D = \overline{\delta}_\lambda(x_0)$ we conclude that $\sum_k r_k^{\overline{\delta}_\lambda(x_0)+\varepsilon''} = \infty$. $\square$

6 Dimension estimates

6.1 Lower bounds for the dimension

Our lower estimate of the dimension is based into the construction of a Cantor like set. Our argument requires to compare the measures λ and μ several times because we use some consequences of the Shannon-McMillan-Breiman Theorem for the measure μ (see Section 3) and also some consequences of the definition of expanding maps involving the measure λ. We have already mentioned that the measures λ and μ are comparable into the blocks of the partition $\mathcal{P}_0$, but in order to control the comparability constants in our proof, we need the following definition:

Definition 6.1. *We will say that a point $x_0 \in X_0$ is* <u>*approximable*</u> *if there exist an increasing sequence $\mathcal{I}(x_0) = \{p_i\}$ of natural numbers such that for all $A \in \mathcal{A}$ contained in $P(0, T^{p_i}(x_0))$ for some i, we have that*

$$\frac{1}{K}\,\lambda(A) \leq \mu(A) \leq K\,\lambda(A)\,.$$

with $K > 1$ a constant depending on x_0.

Remark 6.1. A mixing version of Poincare's Recurrence Theorem (see [19], Theorem A') shows that for λ-almost all point x_0 there exists an increasing sequence $\{p_i\}$ such that $P(0, T^{p_i}(x_0)) = P(0, T^{p_1}(x_0))$ for all i. Therefore, the set of approximable points have full λ-measure.

Remark 6.2. From part (iii) of Theorem E we have that if the partition $\mathcal{P}_0$ is finite then any point in X_0^+ is an approximable point. More generally, from Remark 4.3 we have that if $\inf_{P\in\mathcal{P}_0} \lambda(T(P)) > 0$ and $\sup_{P\in\mathcal{P}_0} \operatorname{diam}(T(P)) < \infty$, then any point in X_0^+ is an approximable point.

The next theorem contains a lower bound for the Hausdorff and the grid Hausdorff dimensions of $\mathcal{W}(U, x_0, \{r_n\})$ with respect to the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. As we mentioned in Section 2, in order to get results for the λ-Hausdorff dimension we need an extra property of regularity. More precisely, we ask $\mathbf{\Pi}$ to be λ-regular (see Definiton 2.5). We recall that any grid on $\mathbf{R}$ is λ-regular.

Theorem 6.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Let $\{r_n\}$ be a non increasing sequence of positive numbers and let U be an open set in X with $\mu(U) > 0$. Then, for all approximable point $x_0 \in X_0$, the grid Hausdorff dimensions of the set*

$$\mathcal{W}(U, x_0, \{r_n\}) = \{x \in U \cap X_0 :\ d(T^n(x), x_0) < r_n \text{ for infinitely many } n\}$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{W}(U,x_0,\{r_n\})) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\mathcal{W}(U,x_0,\{r_n\})) \geq \frac{h_\mu}{h_\mu + \overline{\delta}_\lambda(x_0)\,\overline{\ell}}\,. \tag{37}$$

where $\overline{\ell} = \limsup_{n\to\infty} \frac{1}{n}\log\frac{1}{r_n}$ *and* h_μ *is the entropy of* T *with respect to* μ.

Moreover, for all approximable point $x_0 \in X_0$*, the Hausdorff dimensions of the set* $\mathcal{W}(U,x_0,\{r_n\})$ *verify:*

1. *If the grid* $\mathbf{\Pi}$ *is* λ*-regular then*

$$\mathrm{Dim}_{\lambda}(\mathcal{W}(U,x_0,\{r_n\})) = \mathrm{Dim}_{\mu}(\mathcal{W}(U,x_0,\{r_n\})) \geq \frac{h_\mu}{h_\mu + \overline{\delta}_\lambda(x_0)\,\overline{\ell}}\Big(1 - \frac{\overline{\tau}_\lambda(x_0)\overline{\delta}_\lambda(x_0)\ell^2}{h_\mu^2\log\beta}\Big)\,, \tag{38}$$

2. *If* λ *is a doubling measure verifying that* $\lambda(B(x,r)) \leq C\,r^s$ *for all ball* $B(x,r)$*, then*

$$\mathrm{Dim}_{\lambda}(\mathcal{W}(U,x_0,\{r_n\})) = \mathrm{Dim}_{\mu}(\mathcal{W}(U,x_0,\{r_n\})) \geq 1 - \frac{\overline{\delta}_\lambda(x_0)\overline{\ell}}{s\log\beta}\,. \tag{39}$$

Here β *is the constant appearing in the property* (C) *of expanding maps.*

Remark 6.3. Recall from Remark 4.4 that $0 < h_\mu < \infty$. Recall also that from Remark 6.1 we know that the set of approximable points has full λ-measure, and from Lemma 4.6 we know that all point x_0 in X_1 satisfies $\overline{\delta}_\lambda(x_0) < \infty$ and $\overline{\tau}_\lambda(x_0) = 0$. Therefore since X_1 has full λ-measure we have that Theorem 6.1 holds with $\overline{\tau}_\lambda(x_0) = 0$ for λ-almost all $x_0 \in X$.

Remark 6.4. From Remark 6.2 we have that if $\inf_{P\in\mathcal{P}_0}\lambda(T(P)) > 0$ and $\sup_{P\in\mathcal{P}_0}\mathrm{diam}\,(T(P)) < \infty$ then any point in X_0^+ is an approximable point. Moreover, if $\sup_{P\in\mathcal{P}_0}\mathrm{diam}\,(T(P)) < \infty$ then, by Proposition 4.2,

$$\frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} \asymp \frac{\lambda(T(P(0,T^n(x_0))))}{\lambda(P(0,T^{n+1}(x_0)))} \leq \frac{\lambda(X)}{\lambda(P(0,T^{n+1}(x_0)))}$$

and then $\overline{\tau}_\lambda(x_0) = 0$ for all x_0 such that

$$\log\frac{1}{\lambda(P(0,T^n(x_0)))} = o(n)\,, \qquad \text{as} \qquad n\to\infty\,.$$

First, let us observe that the λ-Hausdorff dimension and μ-Hausdorff dimensions coincide for subsets of $\cup_{P\in\mathcal{P}_0}P$ and, in particular, for subsets of X_0.

Lemma 6.1. *If* $A \in \mathcal{A}$ *is a subset of* $\cup_{P\in\mathcal{P}_0}P$*, then*

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(A) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(A) \qquad \textit{and} \qquad \mathrm{Dim}_{\lambda}(A) = \mathrm{Dim}_{\mu}(A)\,.$$

Proof. We will prove only the equality of grid-dimensions, since the other proof is similar. By properties (A.2) and (A.3) of expanding maps we have for the α-dimensional λ-grid and μ-grid Hausdorff measures that

$$\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A) = \sum_i \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A\cap P_i)\,, \qquad \mathcal{H}^\alpha_{\mathbf{\Pi},\mu}(A) = \sum_i \mathcal{H}^\alpha_{\mathbf{\Pi},\mu}(A\cap P_i)\,.$$

where $\mathcal{P}_0 = \{P_i\}$. As a consequence of part (iii) of Theorem E we get that

$$\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A\cap P_i) \asymp \mathcal{H}^\alpha_{\mathbf{\Pi},\mu}(A\cap P_i)\,,$$

with constants depending on i. Therefore

$$\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^\alpha_{\mathbf{\Pi},\mu}(A) = 0\,.$$

$\square$

Proof of Theorem 6.1. We may assume that $\overline{\delta}_\lambda(x_0)$, $\overline{\tau}_\lambda(x_0)$ and $\overline{\ell}$ are all finite, since otherwise the estimations (37) and (38) are trivial. Since $\mu(U) > 0$, the set U contains a point $x \in X_0$. As U is open, we have that there exists $r > 0$ such that $B(x, r) \subset U$, where by $B(x, r)$ we denote the ball $\{y \in X : d(y, x) < r\}$. Therefore, since $\sup_{P\in\mathcal{P}_n} \operatorname{diam}(P)$ goes to zero as $n \to \infty$, there exists $N_0 \in \mathbf{N}$ such that

$$P(N_0, x) \subset B(x, r) \subset U.$$

Let us write $J_0 = P(N_0, x)$ and let A_0 denote the element of the partition $\mathcal{P}_0$ such that $T^{N_0}(J_0) = A_0$. To get the desired result, it is enough to show (37) and (38) for the set

$$\widetilde{W} = \{x \in J_0 \cap X_0 : d(T^n(x), x_0) < r_n \text{ for infinitely many } n\}.$$

Notice that we can assume that $\lim_{n\to\infty} r_n = 0$. Otherwise, there exists $C > 0$ such that $r_n > C$ for all n, and since μ is ergodic, from Theorem A' in [19], we deduce that for μ-almost every point in J_0,

$$d(T^n(x), x_0) < C \text{ for infinitely many } n\,. \tag{40}$$

Using Remark 4.2 we conclude that (40) holds also for λ-almost every point in J_0 and therefore $\operatorname{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{W}(U, x_0, \{r_n\})) = \operatorname{Dim}_\lambda(\mathcal{W}(U, x_0, \{r_n\})) = 1$. However, in this case more is true, see Corollary 6.1.

To obtain (37) we will construct, for each small $\varepsilon > 0$, a Cantor-like set $\mathcal{C}_\varepsilon \subset \widetilde{W}$ and we will prove using Corollary 2.1 that

$$\operatorname{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{C}_\varepsilon) = \operatorname{Dim}_{\mathbf{\Pi},\mu}(\mathcal{C}_\varepsilon) \geq \frac{h_\mu - 2\varepsilon}{h_\mu + 2\varepsilon + (1+\varepsilon)(\overline{\delta}_\lambda(x_0) + \varepsilon)\,(\overline{\ell} + \varepsilon) + \varepsilon}\,. \tag{41}$$

We construct now the Cantor-like set

$$\mathcal{C}_\varepsilon = \bigcap_{n=0}^{\infty} \bigcup_{J\in\mathcal{J}_n} J$$

as follows: we start with $\mathcal{J}_0 = \{J_0\}$ and we denote by $G_{J_0}^{-1}$ the composition of the N_0 branchs of T^{-1} such that $J_0 = G_{J_0}^{-1}(A_0)$.

Let $\mathcal{I}(x_0) = \{p_i\}$ denote the sequence associated to the approximable point x_0 given by Definition 6.1. Let $s_k = \operatorname{diam}(P(p_k, x_0))$, and for each s_k let $n(k)$ denotes the greatest natural number such that $s_k \leq r_{n(k)}$. We denote by $\mathcal{D}$ the set of these indexes $n(k)$. Since $s_k \to 0$ as $k \to \infty$ and $r_n \to 0$ as $n \to \infty$ by hypothesis, we have that $n(k) \to \infty$ as $k \to \infty$. We will write $\mathcal{D} = \{d_i\}$ with $d_i < d_{i+1}$ for all i.

Notice that if $d \in \mathcal{D}$, then there exists $k(d) \in \mathcal{I}(x_0)$ such that

$$r_{d+1} < \operatorname{diam}(P(k(d), x_0)) \leq r_d \tag{42}$$

and

$$P(k(d), x_0) \subset B(x_0, \operatorname{diam}(P(k(d), x_0)) \subset B(x_0, r_d)\,, \tag{43}$$

Moreover from the property (C) of expanding maps we have that

$$C_2\beta^{k(d)}\operatorname{diam}(P(k(d), x_0)) \leq \operatorname{diam}(P(0, T^{k(d)}(x_0))) \leq \sup_{P\in\mathcal{P}_0} \operatorname{diam}(P) < \infty$$

and using (42) we get

$$\frac{k(d)}{d+1} \leq \frac{1}{\log\beta}\left[\frac{C}{d+1} + \frac{1}{d+1}\log\frac{1}{r_{d+1}}\right].$$

Hence, for all d large enough

$$\frac{k(d)}{d} \leq (1+\varepsilon)\frac{\overline{\ell}}{\log\beta}\,. \tag{44}$$

To construct the family $\mathcal{J}_1$ we first choose a natural number N_1 so that $d_1 := N_0 + N_1 \in \mathcal{D}$, and large enough so that (16) holds with $P_1 = A_0$, $P_2 = P(0, x_0)$, $N = N_1$ and $M := M_1 = [\varepsilon N_0]$, (44) holds for d_1, and also

$$d_1 \leq (1+\varepsilon)N_1\,, \qquad r_{d_1} > e^{-d_1(\overline{\ell}+\varepsilon)}. \tag{45}$$

Let $\mathcal{S}_{N_1,M_1}$ denote the collection of elements in $\mathcal{P}_{N_1}$ given by Proposition 3.1. We define $\widetilde{\mathcal{J}}_1$ as the family of sets $G_{J_0}^{-1}(S)$ with $S \in \mathcal{S}_{N_1,M_1}$, see Figure 1.

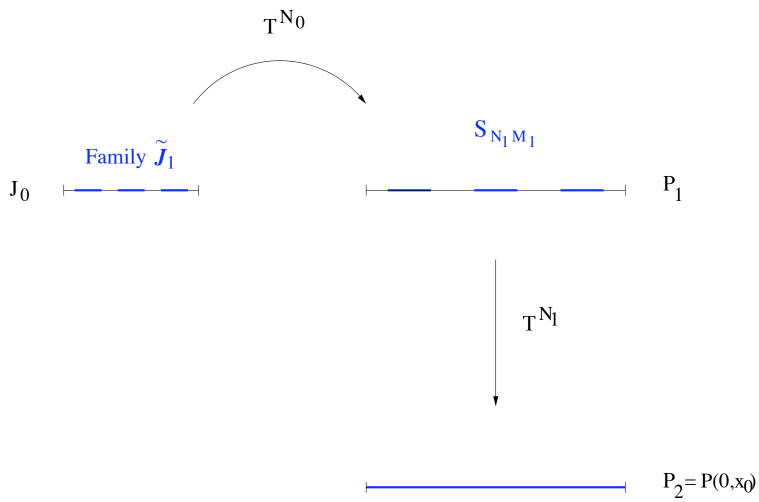

Figure 1.

Notice that by construction if $\widetilde{J} \in \widetilde{\mathcal{J}}_1$, then

$$T^{d_1}(\widetilde{J}) = P(0, x_0), \qquad \text{with} \qquad d_1 = N_0 + N_1 \tag{46}$$

and since $T^{N_0}(\widetilde{J}) = S$ for some $S \in \mathcal{S}_{N_1,M_1}$

$$T^{N_0}(\widetilde{J}) \cap (X_0 \setminus E^{\varepsilon}_{M_1}) \neq \emptyset \qquad \text{with} \qquad M_1 = [\varepsilon N_0].$$

We remark that we will define later the family $\mathcal{J}_1$ by taking an appropriate subset of each one of the elements of the family $\widetilde{\mathcal{J}}_1$. From Proposition 4.2 we obtain that if $\widetilde{J} = G^{-1}_{J_0}(S)$ with $S \in \mathcal{S}_{N_1,M_1}$, then

$$\frac{1}{C}\,\frac{\lambda(S)}{\lambda(A_0)} \leq \frac{\lambda(\widetilde{J})}{\lambda(J_0)} \leq C\,\frac{\lambda(S)}{\lambda(A_0)} \tag{47}$$

with C an absolute constant. Hence, from (14), (47), part (iii) of Theorem E and by taking N_1 large enough we get that for all $\widetilde{J}_1 \in \widetilde{\mathcal{J}}_1$

$$e^{-N_1(h_\mu + 2\varepsilon)} \leq \frac{\lambda(\widetilde{J}_1)}{\lambda(J_0)} \leq e^{-N_1(h_\mu - 2\varepsilon)}\,,$$

From (47) we also get an estimate on the size of the family $\widetilde{\mathcal{J}}_1$

$$\lambda(\widetilde{\mathcal{J}}_1 \cap J_0) = \lambda(\widetilde{\mathcal{J}}_1) := \sum_{\widetilde{J}_1 \in \widetilde{\mathcal{J}}_1} \lambda(\widetilde{J}_1) \geq \frac{1}{C}\,\frac{\lambda(J_0)}{\lambda(A_0)} \sum_{S \in \mathcal{S}_{N_1,M_1}} \lambda(S)\,.$$

Therefore, from Proposition 3.1, and the part (iii) of Theorem E we get that

$$\lambda(\widetilde{\mathcal{J}}_1 \cap J_0) \geq C\,\lambda(J_0)\,\lambda(P(0, x_0))$$

with $C > 0$ depending on $P_1 = A_0$ and $P_2 = P(0.x_0)$.

Now, since $d_1 = N_0 + N_1 \in \mathcal{D}$, by (42) and (43) there exists an integer $k_1 \in \mathcal{I}(x_0)$ such that

$$P(k_1, x_0) \subset B(x_0,\, r_{d_1}). \tag{48}$$

and

$$\operatorname{diam}(P(k_1, x_0)) \geq r_{d_1+1}\,. \tag{49}$$

Moreover, from (45) we have that $k_1 \leq \frac{(1+\varepsilon)^2\bar{\ell}}{\log\beta} N_1$. Since we can take N_1 as large as we want so that k_1 is large enough we can get that

$$\text{closure}\,(P(k_1, x_0)) \subset P(0, x_0) \tag{50}$$

and also by taking N_1 large we have from the definition of $\overline{\delta}_\lambda(x_0)$ and (49) and (45) that

$$\lambda(P(k_1, x_0)) \geq \mathrm{diam}(P(k_1, x_0))^{\overline{\delta}_\lambda(x_0)+\varepsilon} \geq r_{d_1+1}^{\overline{\delta}_\lambda(x_0)+\varepsilon} \geq e^{-N_1(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)}. \tag{51}$$

For all $S \in \mathcal{S}_{N_1,M_1}$ let G_S^{-1} denote the composition of the N_1 branchs of T^{-1} such that $S = G_S^{-1}(P(0, x_0))$. In each set S in $\mathcal{S}_{N_1,M_1}$ we take the subset $L_1 := G_S^{-1}(P(k_1, x_0))$ and we denote by $\mathcal{L}_1$ this family of sets. To define the family $\mathcal{J}_1$ we just "draw" the sets L_1 in J_0. More precisely, $\mathcal{J}_1$ is the family $G_{J_0}^{-1}(L_1)$ with $L_1 \in \mathcal{L}_1$, see Figure 2.

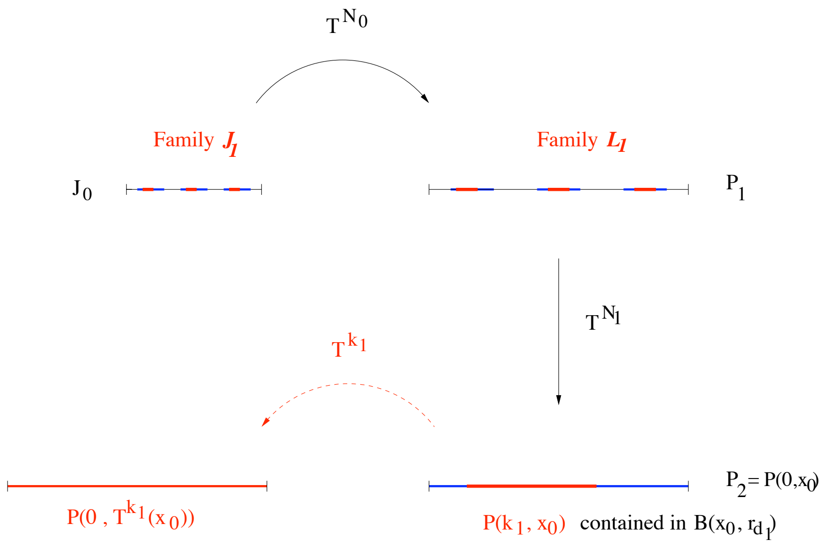

Figure 2.

Notice that by construction if $J \in \mathcal{J}_1$, then

$$T^{d_1}(J) = P(k_1, x_0) \qquad \text{and} \qquad T^{d_1+k_1}(J) = P(0, T^{k_1}(x_0)). \tag{52}$$

Hence if $x \in J \in \mathcal{J}_1$ then $T^{d_1}(x) \in P(k_1, x_0) \subset B(x_0, r_{d_1})$, and it follows that

$$d(T^{d_1}(x), x_0) \leq r_{d_1}\,.$$

By construction we have that for all $J \in \mathcal{J}_1$ there exists an unique $\widetilde{J} \in \widetilde{\mathcal{J}}_1$ such that $J \subset \widetilde{J}$, and by using the condition (A.6) and (50) we have that

$$\mathrm{closure}(J_1) = G_{J_0}^{-1}(G_S^{-1}(\mathrm{closure}(P(k_1, x_0)))) \subset G_{J_0}^{-1}(G_S^{-1}((P(0, x_0))) = \widetilde{J}_1\,.$$

Also by (46), (52) and Proposition 4.2 we get that

$$\frac{1}{C}\,\lambda(P(k_1, x_0)) \leq \frac{\lambda(J_1)}{\lambda(\widetilde{J}_1)} \leq C\lambda(P(k_1, x_0))$$

with $C > 0$ a constant depending on $\lambda(P(0, x_0))$. And from (51) by taking N_1 large, we have that

$$\frac{\lambda(J_1)}{\lambda(\widetilde{J}_1)} \geq e^{-N_1[(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon]}.$$

Now, let us assume that we have already constructed the families $\widetilde{\mathcal{J}}_j$, $\mathcal{J}_j$ and the numbers N_j and $k_j \in \mathcal{I}(x_0)$ for $j = 1, \ldots, m$ with the following properties:

Let $d_1 = N_0 + N_1$ and

$$d_j := N_0 + N_1 + \cdots + N_j + k_1 + \cdots + k_{j-1} \qquad \text{for} \qquad j \geq 2$$

.

(a) For all point x in $J_j \in \mathcal{J}_j$

$$d(T^{d_j}(x), x_0) \leq r_{d_j}.$$

(b) For all $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ we have

(b1) $T^{d_j}(\widetilde{J}_j) = P(0, x_0)$ and

$$T^{d_j - N_j}(\widetilde{J}_j) \bigcap (X_0 \setminus E^{\varepsilon}_{M_j}) \neq \emptyset \qquad \text{with} \qquad M_j = [\varepsilon N_{j-1}].$$

(b2) There exists a unique $J_{j-1} \in \mathcal{J}_{j-1}$ so that $\widetilde{J}_j \subset J_{j-1}$ and

$$e^{-N_j(h_\mu + 2\varepsilon)} \leq \frac{\lambda(\widetilde{J}_j)}{\lambda(J_{j-1})} \leq e^{-N_j(h_\mu - 2\varepsilon)}.$$

(c) For all $J_j \in \mathcal{J}_j$ we have

(c1) $T^{d_j}(J_j) = P(k_j, x_0)$.

(c2) There exists a unique $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ so that closure$(J_j) \subset \widetilde{J}_j$,

$$\frac{\lambda(J_j)}{\lambda(\widetilde{J}_j)} \asymp \lambda(P(k_j, x_0)) \qquad \text{and} \qquad \frac{\lambda(J_j)}{\lambda(\widetilde{J}_j)} \geq e^{-N_j[(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon]}.$$

Besides, for each $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ there exists a unique $J_j \in \mathcal{J}_j$ so that closure$(J_j) \subset \widetilde{J}_j$

(c3) $k_j \leq \frac{(1+\varepsilon)^2 \overline{\ell}}{\log \beta} N_j$.

(d) There exists an absolute constant $\widetilde{c} > 1$ such that for all $J_{j-1} \in \mathcal{J}_{j-1}$,

$$\lambda(\widetilde{\mathcal{J}}_j \cap J_{j-1}) := \sum_{\widetilde{J}_j \in \widetilde{\mathcal{J}}_j,\, \widetilde{J}_j \subset J_{j-1}} \lambda(\widetilde{J}_j) \geq \frac{1}{\widetilde{c}}\, \lambda(J_{j-1}).$$

(e) N_j is big enough so that

$$\frac{j}{N_1 + \cdots + N_j} < \frac{1}{j}.$$

We want to mention that the hypothesis on x_0 of being approximable is only required to obtain an absolute constant $\widetilde{c}$ in the property (d).

Recall that we want to apply Corollary 2.1. In our case

$$a = h_\mu + 2\varepsilon, \qquad b = h_\mu - 2\varepsilon, \qquad c = (1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon \qquad \text{and} \qquad \delta = 1/\widetilde{c}. \quad (53)$$

Now we start with the construction of the family $\mathcal{J}_{m+1}$. We choose a natural number $N_{m+1} \geq N_m$ large enough so that

$$d_{m+1} := N_0 + N_1 + \cdots + N_{m+1} + k_1 + \cdots + k_m \in \mathcal{D},$$

property (e) holds for $j = m+1$, (16) holds with $P_1 = P(0, T^{k_m}(x_0))$, $P_2 = P(0, x_0)$, $N = N_{m+1}$, and $M := M_{m+1} = [\varepsilon N_m]$, (44) holds for d_{m+1}, and also

$$d_{m+1} \leq (1+\varepsilon) N_{m+1}, \qquad r_{d_{m+1}} \geq e^{-d_{m+1}(\overline{\ell}+\varepsilon)}. \quad (54)$$

Let $\mathcal{S}_{N_{m+1}, M_{m+1}}$ denote the collection of elements in $\mathcal{P}_{N_{m+1}}$ given by Proposition 3.1. Notice that the sets in this family verify (14) with $N = N_{m+1}$. For each $J \in \mathcal{J}_m$ let G_J^{-1} denote the composition of the $d_m + k_m$ branchs of T^{-1} such that $J = G_J^{-1}(P(0, T^{k_m}(x_0)))$. We define now $\widetilde{\mathcal{J}}_{m+1}$ as

$$\widetilde{\mathcal{J}}_{m+1} = \bigcup_{J \in \mathcal{J}_m} G_J^{-1}(\mathcal{S}_{N_{m+1}, M_{m+1}}).$$

Notice that, by construction, if $\widetilde{J} \in \widetilde{\mathcal{J}}_{m+1}$, then

$$T^{d_{m+1}}(\widetilde{J}) = P(0, x_0), \quad (55)$$

and since $T^{d_{m+1} - N_{m+1}}(\widetilde{J}) = S$ for some $S \in \mathcal{S}_{N_{m+1}, M_{m+1}}$.

$$T^{d_{m+1} - N_{m+1}}(\widetilde{J}) \cup (X_0 \setminus E^{\varepsilon}_{M_{m+1}}) \neq \emptyset \qquad \text{with} \qquad M_{m+1} = [\varepsilon N_m].$$

Since $J_m \in \mathcal{P}_{d_m+k_m}$, by Proposition 4.2 we have that if $\widetilde{J} = G_{J_m}^{-1}(S)$ with $S \in \mathcal{S}_{N_{m+1},M_{m+1}}$, then

$$\frac{1}{C}\,\frac{\lambda(S)}{\lambda(P(0,T^{k_m}(x_0)))} \le \frac{\lambda(\widetilde{J})}{\lambda(J_m)} \le C\,\frac{\lambda(S)}{\lambda(P(0,T^{k_m}(x_0)))}\,, \tag{56}$$

with C an absolute constant.

If $\widetilde{J}_{m+1} \in \widetilde{\mathcal{J}}_{m+1}$, then there are $J_m \in \mathcal{J}_m$ and $S \in \mathcal{S}_{N_{m+1},M_{m+1}}$ such that $\widetilde{J}_{m+1} = G_{J_m}^{-1}(S)$. Hence from (56), (14), the definition of approximable points and by taking N_{m+1} large, we get the property (b) for $j = m+1$. We remark that $\lambda(P(0,T^{k_m}(x_0))$ does not depend on N_{m+1}.

Now from (56), Proposition 3.1 and again the definition of approximable points we get

$$\lambda(\widetilde{\mathcal{J}}_{m+1} \cap J_m) = \sum_{\substack{\widetilde{J}_{m+1} \in \widetilde{\mathcal{J}}_{m+1} \\ \widetilde{J}_{m+1} \subset J_m}} \lambda(\widetilde{J}_{m+1}) \ge \frac{1}{C}\,\frac{\lambda(J_m)}{\lambda(P(0,T^{k_m}(x_0)))} \sum_{S \in \mathcal{S}_{N_{m+1},M_{m+1}}} \lambda(S) \ge \frac{1}{c'}\,\lambda(P(0,x_0))\,\lambda(J_m)\,,$$

and this gives property (d) for $j = m+1$. Observe that the constant c' depends on the comparability constant between λ and μ in $P_1 = P(0,T^{k_m}(x_0))$ but from the definition of approximable points we know that this constant is absolute.

Since $d_{m+1} \in \mathcal{D}$, by (42) and (43), there exists an integer $k_{m+1} \in \mathcal{I}(x_0)$ such that

$$P(k_{m+1},x_0) \subset B(x_0, r_{d_{m+1}}). \tag{57}$$

and

$$\operatorname{diam}(P(k_{m+1},x_0)) \ge r_{d_{m+1}+1}\,. \tag{58}$$

From (44) and since $d_{m+1} \le (1+\varepsilon)N_{m+1}$ we get the property (c3) for $j = m+1$.

As in the initial step from the definition of $\overline{\delta}_\lambda(x_0)$, (58) and (54), we have that

$$\lambda(P(k_{m+1},x_0)) \ge r_{d_{m+1}+1}^{\overline{\delta}_\lambda(x_0)+\varepsilon} \ge e^{-N_{m+1}(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)}. \tag{59}$$

In each set $S \in \mathcal{S}_{N_{m+1},M_{m+1}}$ we take the subset $L_{m+1} := G_S^{-1}(P(k_{m+1},x_0))$ and we call $\mathcal{L}_{m+1}$ to this family of sets. We recall that by G_S^{-1} we denote the composition of the N_{m+1} branchs of T^{-1} such that $S = G_S^{-1}(P(0,x_0))$.

To define the family $\mathcal{J}_{m+1}$ we "draw" the family $\mathcal{L}_{m+1}$ in each one of the sets $J \in \mathcal{J}_m$. More precisely, for each $J \in \mathcal{J}_m$ let G_J denote the composition of the $d_m + k_m$ branchs of T such that $G_J(J) = P(0,T^{k_m}(x_0))$. We define now $\mathcal{J}_{m+1}$ as

$$\mathcal{J}_{m+1} = \bigcup_{J \in \mathcal{J}_m} G_J^{-1}(\mathcal{L}_{m+1})\,.$$

Notice that by construction if $J \in \mathcal{J}_{m+1}$, then

$$T^{d_{m+1}}(J) = P(k_{m+1},x_0) \qquad \text{and} \qquad T^{d_{m+1}+k_{m+1}}(J) = P(0,T^{k_{m+1}}(x_0)). \tag{60}$$

Therefore the condition (c1) holds for $j = m+1$. Besides, by (57), if $x \in J \in \mathcal{J}_{m+1}$ then $T^{d_{m+1}}(x) \in P(k_{m+1},x_0) \subset B(x_0, r_{d_{m+1}})$, and therefore the condition (a) holds for $j = m+1$.

By construction we have that for all $J_{m+1} \in \mathcal{J}_{m+1}$ there exists an unique $\widetilde{J}_{m+1} \in \widetilde{\mathcal{J}}_{m+1}$ such that $J_{m+1} \subset \widetilde{J}_{m+1}$, and by using the condition (A.6) and (50) as in the initial step we have that

$$\operatorname{closure}(J_{m+1}) \subset \widetilde{J}_{m+1}\,.$$

The estimates of $\lambda(J_{m+1})/\lambda(\widetilde{J}_{m+1})$ of the condition (c2) follows by applying Proposition 4.2 and by using (55), (59) and (60).

We have already obtained the properties (a)-(e) for $j = m+1$, and therefore we have concluded the construction of the Cantor-like set $\mathcal{C}_\varepsilon$. The property that for all $J_{m+1} \in \mathcal{J}_{m+1}$ there exists a unique $J_m \in \mathcal{J}_m$ such that

$$\operatorname{closure}(J_{m+1}) \subset J_m$$

implies that $\mathcal{C}_\varepsilon$ is not empty. And moreover $\mathcal{C}_\varepsilon \subset X_0$ since by construction $P(m,x)$ is defined for all $x \in \mathcal{C}_\varepsilon$. Hence the condition (a) implies that $\mathcal{C}_\varepsilon$ is contained in the set W.

The estimate (41) follows now directly from (53), property (e) and Corollary 2.1.

Next we will prove the estimate (38) for the λ-grid Hausdorff dimension of $\mathcal{C}_\varepsilon$. We will use the subcollections $\{\mathcal{Q}_m\}$ of $\{\mathcal{P}_m\}$ given by

$$\mathcal{Q}_m = \{P(m,x) \,:\, x \in \mathcal{C}_\varepsilon\}$$

in order to apply Proposition 2.1. Since $\mathbf{\Pi}$ is a λ-regular grid, see Definition 2.5, we only need to deal with the computation of the parameters $\{a_m\}$ and $\{b_m\}$ of the subcolletions $\mathcal{Q}_m$. We recall that a_m and b_m are, respectively, a lower and an upper bound for $\lambda(P(m,x))$ with $x \in \mathcal{C}_\varepsilon$.

The easiest cases correspond to $m = d_n$ and $m = d_n + k_n$, i.e. to the families $\widetilde{\mathcal{J}}_n$ and $\mathcal{J}_n$. Since $P(d_n, x)$ belongs to $\widetilde{\mathcal{J}}_n$, from property (b2) of $\mathcal{C}_\varepsilon$ and by taking N_n large enough we have that

$$a_{d_n} = e^{-N_n(h_\mu+3\varepsilon)} \qquad \text{and} \qquad b_{d_n} = e^{-N_n(h_\mu-3\varepsilon)}\,. \tag{61}$$

Also, from property (c2) for $j = n$,

$$\lambda(P(d_n+k_n,x)) \asymp \lambda(P(d_n,x))\,\lambda(P(k_n,x_0))\,.$$

for all $x \in \mathcal{C}_\varepsilon$, and therefore,

$$a_{d_n+k_n} \asymp a_{d_n}\lambda(P(k_n,x_0)) \qquad \text{and} \qquad b_{d_n+k_n} \asymp b_{d_n}\lambda(P(k_n,x_0))\,. \tag{62}$$

To estimate a_m and b_m in the other cases we need first some estimate on the Jacobian. Specifically we need to estimate $\mathbf{J}_{d_n}(x)$ and $\mathbf{J}_{d_n+k_n}(x)$ for $x \in \mathcal{C}_\varepsilon$. From (17), Proposition 4.1, and properties (b1) and (c1) of $\mathcal{C}_\varepsilon$ (for $j = n$) we have that

$$\lambda(P(0,x_0)) = \lambda(T^{d_n}(P(d_n,x))) = \int_{P(d_n,x)} \mathbf{J}_{d_n}\,d\lambda \asymp \mathbf{J}_{d_n}(x)\,\lambda(P(d_n,x))$$

and

$$\lambda(P(0,T^{k_n}(x_0))) = \lambda(T^{d_n+k_n}(P(d_n{+}k_n,x))) = \int_{P(d_n+k_n,x)} \mathbf{J}_{d_n+k_n}\,d\lambda \asymp \mathbf{J}_{d_n+k_n}(x)\,\lambda(P(d_n{+}k_n,x))\,.$$

Hence for all $x \in \mathcal{C}_\varepsilon$

$$\frac{1}{\mathbf{J}_{d_n}(x)} \asymp \lambda(P(d_n,x)) \tag{63}$$

and, by using property (c2) for $j = n$,

$$\frac{1}{\mathbf{J}_{d_n+k_n}(x)} \asymp \frac{\lambda(P(d_n,x))\,\lambda(P(k_n,x_0))}{\lambda(P(0,T^{k_n}(x_0)))} \tag{64}$$

with constants depending on $P(0,x_0)$.

For $d_n < m < d_n + k_n$ by (17) and Proposition 4.1 we have that

$$\lambda(P(m-d_n,T^{d_n}(x))) = \lambda(T^{d_n}(P(m,x))) = \int_{P(m,x)} \mathbf{J}_{d_n}\,d\lambda \asymp \mathbf{J}_{d_n}(x)\,\lambda(P(m,x))\,.$$

Then from (63) we get

$$\lambda(P(m,x)) \asymp \lambda(P(d_n,x))\,\lambda(P(m-d_n,T^{d_n}(x)))\,.$$

But, since $x \in \mathcal{C}_\varepsilon$, by (c1)

$$T^{d_n}(x) \in P(k_n,x_0) \subseteq P(m-d_n,x_0), \qquad \text{for } m \le d_n + k_n,$$

and therefore $P(m-d_n,T^{d_n}(x)) = P(m-d_n,x_0)$. Hence we have that for $d_n \le m \le d_n + k_n$

$$a_m \asymp a_{d_n}\lambda(P(m-d_n,x_0)) \qquad \text{and} \qquad b_m \asymp b_{d_n}\lambda(P(m-d_n,x_0))\,. \tag{65}$$

Now for $d_n + k_n < m < d_{n+1} = d_n + k_n + N_{n+1}$ by (17) and Proposition 4.1 we have that

$$\lambda(T^{d_n+k_n}(P(m,x))) = \int_{P(m,x)} \mathbf{J}_{d_n+k_n}\,d\lambda \asymp \mathbf{J}_{d_n+k_n}(x)\,\lambda(P(m,x))$$

and from (64) we get

$$\lambda(P(m,x)) \asymp \frac{\lambda(P(d_n,x))\,\lambda(P(k_n,x_0)\,\lambda(T^{d_n+k_n}(P(m,x)))}{\lambda(P(0,T^{k_n}(x_0)))}\,. \tag{66}$$

Therefore, we need to obtain upper and lower bounds of $\lambda(T^{d_n+k_n}(P(m,x)))$ independent of $x \in \mathcal{C}_\varepsilon$.

Notice that if $m \le d_{n+1}$, then

$$T^{d_{n+1}-N_{n+1}}(P(d_{n+1},x)) = T^{d_n+k_n}(P(d_{n+1},x)) \subset T^{d_n+k_n}(P(m,x))$$

and, since $T^{d_{n+1}-N_{n+1}}(P(d_{n+1},x)) = P(N_{n+1}, T^{d_n+k_n}(x))$ is an element of the family $\mathcal{S}_{N_{n+1},M_{n+1}}$, from the property (b1) of $\mathcal{C}_\varepsilon$ we can conclude that there exists $z \in T^{d_n+k_n}(P(m,x))$ such that $z \notin E^\varepsilon_{M_{n+1}}$ with $M_{n+1} = [\varepsilon N_n]$. Hence, for $m \le d_{n+1}$,

$$T^{d_n+k_n}(P(m,x)) = P(m-d_n-k_n, z) \qquad \text{with} \qquad z \notin E^\varepsilon_{[\varepsilon N_n]} \tag{67}$$

and, for $m - d_n - k_n \ge M_{n+1} = [\varepsilon N_n]$,

$$\frac{1}{C} e^{-(m-d_n-k_n)(h_\mu+\varepsilon)} \le \lambda(T^{d_n+k_n}(P(m,x))) \le C e^{-(m-d_n-k_n)(h_\mu-\varepsilon)}\,.$$

Therefore, for $d_n + k_n + [\varepsilon N_n] \le m < d_{n+1}$,

$$a_m \asymp \frac{a_{d_n}\lambda(P(k_n,x_0))e^{-(m-d_n-k_n)(h_\mu+\varepsilon)}}{\lambda(P(0,T^{k_n}(x_0)))} \quad \text{and} \quad b_m \asymp \frac{b_{d_n}\lambda(P(k_n,x_0))e^{-(m-d_n-k_n)(h_\mu-\varepsilon)}}{\lambda(P(0,T^{k_n}(x_0)))}\,. \tag{68}$$

For $d_n + k_n < m < d_n + k_n + [\varepsilon N_n]$ we have that

$$P([\varepsilon N_n], z) \subset T^{d_n+k_n}(P(m,x)) \subset P(0, T^{k_n}(x_0))$$

and therefore, from Lemma 3.1 and the definition of approximable point (recall that $k_n \in \mathcal{I}(x_0)$),

$$\frac{1}{C}\, e^{-[\varepsilon N_n](h_\mu+\varepsilon)} \le \lambda(T^{d_n+k_n}(P(m,x))) \le \lambda(P(0,T^{k_n}(x_0))). \tag{69}$$

Hence, from (66) we get that

$$a_m \asymp \frac{a_{d_n}\lambda(P(k_n,x_0))e^{-[\varepsilon N_n](h_\mu+\varepsilon)}}{\lambda(P(0,T^{k_n}(x_0)))} \qquad \text{and} \qquad b_m \asymp b_{d_n}\lambda(P(k_n,x_0))\,. \tag{70}$$

In order to apply Proposition 2.1 we will show that

$$\limsup_{m\to\infty} \frac{\log(1/a_m)}{\log(1/b_{m-1})} \le 1 + C\varepsilon + \frac{\overline{\tau}_\lambda(x_0)\overline{\ell}}{(h_\mu - 3\varepsilon)\log\beta} \tag{71}$$

with C an absolute constant. Then, from Proposition 2.1, (71), and by taking $\varepsilon \to 0$ we get the desired bound for the λ-Hausdorff dimension of the set $\mathcal{W}(U, x_0, \{r_n\})$.

Let us define

$$q_m := \frac{\log(1/a_m)}{\log(1/b_{m-1})}\,.$$

For $m = d_n$ we have by (61) and (68) that

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon)}{N_n(h_\mu - \varepsilon) + C_{n-1}} \tag{72}$$

with

$$C_{n-1} = (h_\mu - 3\varepsilon)N_{n-1} + \log\frac{\lambda(P(0,T^{k_{n-1}}(x_0)))}{\lambda(P(k_{n-1},x_0))}\,.$$

Hence from (72) we get that

$$q_m \le 1 + C\varepsilon \qquad \text{for} \qquad m = d_n \tag{73}$$

with $C > 0$ an absolute constant. In order to obtain (71) from (72) we must also say that we have taken N_n large enough so that $C_{n-1} \ge -\varepsilon N_n$.

For $m = d_n + 1$ we get from (61) and (65) that, for N_n large enough,

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon) + \log \frac{1}{\lambda(P(1,x_0))}}{N_n(h_\mu - 3\varepsilon)} \le 1 + C\varepsilon\,. \tag{74}$$

For $d_n + 1 < m \le d_n + k_n$ we have from (65)

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon) + \log \frac{1}{\lambda(P(m-d_n,x_0))}}{N_n(h_\mu - 3\varepsilon) + \log \frac{1}{\lambda(P(m-d_n-1,x_0))}}\,. \tag{75}$$

But since

$$\overline{\tau}_\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))},$$

then given $\varepsilon > 0$ there exists $C' > 0$ such that for all n

$$\log \frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} \le C' + n(\overline{\tau}_\lambda(x_0) + \varepsilon)\,.$$

Hence

$$\log \frac{\lambda(P(m-d_n-1,x_0))}{\lambda(P(m-d_n,x_0))} \le C' + k_n(\overline{\tau}_\lambda(x_0) + \varepsilon)$$

and by property (c3) of $\mathcal{C}_\varepsilon$ for N_n large enough we get that

$$\log \frac{\lambda(P(m-d_n-1,x_0))}{\lambda(P(m-d_n,x_0))} \le C' + \frac{(\overline{\tau}_\lambda(x_0)+\varepsilon)(1+\varepsilon)^2\overline{\ell}}{\log\beta} N_n. \tag{76}$$

From (74), (75) and (76) it follows that for N_n large

$$q_m \le 1 + C\varepsilon + \frac{\overline{\tau}_\lambda(x_0)\overline{\ell}}{(h_\mu - 3\varepsilon)\log\beta} \qquad \text{for} \quad d_n < m \le d_n + k_n \tag{77}$$

with $C > 0$ an absolute constant.

For $d_n + k_n < m < d_n + k_n + [\varepsilon N_n]$ we get from (62), (68) and (70) that

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon) + [\varepsilon N_n](h_\mu + \varepsilon) + \log \frac{1}{\lambda(P(k_n,x_0))} - \log \frac{1}{\lambda(P(0,T^{k_n}(x_0)))}}{N_n(h_\mu - 3\varepsilon) + \log \frac{1}{\lambda(P(k_n,x_0))}}$$

and therefore, for N_n large enough, we have with C an absolute constant that

$$q_m \le 1 + \frac{6\varepsilon N_n + [\varepsilon N_n](h_\mu + \varepsilon)}{N_n(h_\mu - 3\varepsilon)} \le 1 + C\varepsilon \qquad \text{for} \quad d_n + k_n < m < d_n + k_n + [\varepsilon N_n]\,. \tag{78}$$

For $m = d_n + k_m + [\varepsilon N_n]$ we get from (70) and (68) that

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon) + [\varepsilon N_n](h_\mu + \varepsilon) + \log \frac{1}{P(k_n,x_0)} - \log \frac{1}{\lambda(P(0,T^{k_n}(x_0)))}}{N_n(h_\mu - 3\varepsilon) + \log \frac{1}{\lambda(P(k_n,x_0))}}$$

and therefore, for N_n large enough, we have, with C an absolute constant, that

$$q_m \le \frac{6\varepsilon N_n + [\varepsilon N_n](h_\mu + \varepsilon)}{N_n(h_\mu - 3\varepsilon)} \le 1 + C\varepsilon \qquad \text{for} \quad m = d_n + k_n + [\varepsilon N_n]\,. \tag{79}$$

For $d_n + k_n + [\varepsilon N_n] < m < d_{n+1}$ we get from (68) that

$$q_m \asymp \frac{N_n(h_\mu + 3\varepsilon) + (m - d_n - k_n)(h_\mu + \varepsilon) + \log \frac{1}{P(k_n,x_0)} - \log \frac{1}{\lambda(P(0,T^{k_n}(x_0)))}}{N_n(h_\mu - 3\varepsilon) + (m - d_n - k_n - 1)(h_\mu + \varepsilon) + \log \frac{1}{P(k_n,x_0)} - \log \frac{1}{\lambda(P(0,T^{k_n}(x_0)))}}\,. \tag{80}$$

Hence from (69) and (80) we have that

$$q_m \le 1 + \frac{6\varepsilon N_n + h_\mu + \varepsilon}{N_n(h_\mu - 3\varepsilon) - \log \frac{1}{\lambda(P(0,T^{k_n}(x_0)))}} \le 1 + \frac{6\varepsilon N_n + h_\mu + \varepsilon}{N_n(h_\mu - 3\varepsilon) + \log \frac{1}{C} - [\varepsilon N_n](h_\mu + \varepsilon)}$$

and so, for N_n large enough, we have, with C an absolute constant, that

$$q_m \le 1 + C\varepsilon \qquad \text{for} \qquad d_n + k_n + [\varepsilon N_n] < m < d_{n+1}\,. \tag{81}$$

From (73), (77), (78), (79) and (81) we get (71). Using now Proposition 2.1 it follows that

$$\frac{1-\mathrm{Dim}_\lambda(\mathcal{C}_\varepsilon)}{1-\mathrm{Dim}_{\Pi,\lambda}(\mathcal{C}_\varepsilon)} \le 1 + C\varepsilon + \frac{\overline{\tau}_\lambda(x_0)\overline{\ell}}{(h_\mu - 3\varepsilon)\log\beta}\,. \tag{82}$$

As $\mathcal{C}_\varepsilon \subset \mathcal{W}(U, x_0, \{r_n\})$ for all $\varepsilon > 0$, (38) follows now from (41) and (82) by letting $\varepsilon \to 0$.

Finally, to prove (39) it is enough to show that, for all $x \in \mathcal{C}_\varepsilon$,

$$\nu(B(x,r)) \le C\,(\lambda(B(x,r))^\eta\,, \qquad \text{with } \eta = 1 - (1+\varepsilon)\frac{(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon}{s\log\beta}\,. \tag{83}$$

First, notice that from the definition of the measure ν and the properties (c2) and (d) of the definition of the Cantor set $\mathcal{C}_\varepsilon$ it follows that for all $x \in \mathcal{C}_\varepsilon$:

(1) If $J_{n+1} \subset P(m,x) \subseteq \widetilde{J}_{n+1}$, then

$$\nu(P(m,x)) = \nu(\widetilde{J}_{n+1}) = \nu(J_{n+1}) \le C\lambda(\widetilde{J}_{n+1})\,\frac{\nu(J_n)}{\lambda(J_n)} \le C\,\frac{1}{\lambda(P(k_{n+1},x_0))}\,\frac{\nu(J_n)}{\lambda(J_n)}\,\lambda(P(m,x))\,.$$

(2) If $\widetilde{J}_{n+1} \subset P(m,x) \subseteq J_n$, then

$$\begin{aligned}
\nu(P(m,x)) &= \sum_{\substack{\tilde{J}_{n+1}\in\widetilde{\mathcal{J}}_{n+1}\\ \tilde{J}_{n+1}\subseteq P(m,x)}} \nu(J_{n+1}) \;=\; \sum_{\substack{\tilde{J}_{n+1}\in\widetilde{\mathcal{J}}_{n+1}\\ \tilde{J}_{n+1}\subseteq P(m,x)}} \frac{\lambda(\widetilde{J}_{n+1})}{\lambda(\widetilde{\mathcal{J}}_{n+1}\cap J_n)}\,\nu(J_n)\\
&\le C\,\frac{\nu(J_n)}{\lambda(J_n)} \sum_{\substack{\tilde{J}_{n+1}\in\widetilde{\mathcal{J}}_{n+1}\\ \tilde{J}_{n+1}\subseteq P(m,x)}} \lambda(\widetilde{J}_{n+1}) \le C\,\frac{\nu(J_n)}{\lambda(J_n)}\,\lambda(P(m,x))\\
&\le C\,\frac{\lambda(\widetilde{J}_n)\,\nu(J_{n-1})}{\lambda(J_n)\,\lambda(\widetilde{\mathcal{J}}_n\cap J_{n-1})}\,\lambda(P(m,x)) \le C\,\frac{\lambda(\widetilde{J}_n)}{\lambda(J_n)}\,\frac{\nu(J_{n-1})}{\lambda(J_{n-1})}\,\lambda(P(m,x))\\
&\le C\,\frac{1}{\lambda(P(k_n,x_0))}\,\frac{\nu(J_{n-1})}{\lambda(J_{n-1})}\,\lambda(P(m,x))\,.
\end{aligned}$$

In any case, by taking N_{n+1} large enough (and therefore k_{n+1} also large enough) or N_n large enough (and therefore k_n also large enough) we have that for $J_{n+1} \subset P(m,x) \subseteq J_n$,

$$\nu(P(m,x)) \le C\,\frac{1}{\lambda(P(k_j,x_0))^{1+\varepsilon}}\,\lambda(P(m,x)) \tag{84}$$

with $j = n+1$ in the case (1) and $j = n$ in the case (2).

Recall also that by the property (C) of expanding maps we have

$$\sup_{P\in\mathcal{P}_n} \mathrm{diam}(P) \le C\,\frac{1}{\beta^n}$$

Now given a ball $B = B(x,r)$ with center $x \in \mathcal{C}_\varepsilon$ we define the natural number $m = m(B)$ given by

$$\frac{2C}{\beta^m} \le \mathrm{diam}(B) < \frac{2C}{\beta^{m-1}} \tag{85}$$

and the family $\mathcal{P}(B)$ as the collection of blocks in $P \in \mathcal{P}_m$ such that $P \cap B \ne \emptyset$. Let us also denote by $n = n(B)$ the natural number such that $d_n + k_n \le m < d_{n+1} + k_{n+1}$. It is clear that

$$\nu(B) \le \sum_{P\in\mathcal{P}(B)} \nu(P)$$

and using (84) we obtain that

$$\nu(B) \le C\frac{1}{\lambda(P(k_j,x_0))^{1+\varepsilon}} \sum_{P\in\mathcal{P}(B)} \lambda(P)$$

where $j = n$ if $d_n + k_n \le m < d_{n+1}$ and $j = n+1$ if $d_{n+1} \le m < d_{n+1} + k_{n+1}$. Notice that, by (85), it is clear that $\cup_{P\in\mathcal{P}(B)} P \subset 2B := B(x, 2r)$ and since λ is a doubling measure we have that

$$\sum_{P\in\mathcal{P}(B)} \lambda(P) \le C\lambda(B)\,.$$

Therefore, in each of the above cases, we have

$$\nu(B) \le C\,\frac{1}{\lambda(P(k_j, x_0))^{1+\varepsilon}}\,\lambda(B)\,. \tag{86}$$

But, by the property (c2) of the Cantor set $\mathcal{C}_\varepsilon$

$$\lambda(P(k_j, x_0) \ge e^{-N_j[(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon]}$$

and by (85) we obtain that

$$\lambda(B) \le C\,\mathrm{diam}(B)^s \le C\,\frac{1}{\beta^{(m-1)s}} \le C\,e^{-N_j s\log\beta}$$

where we have used that $d_{n+1} \asymp N_{n+1}$ and $d_n + k_n \asymp d_n \asymp N_n$. Hence,

$$\lambda(P(k_j, x_0) \ge \lambda(B)^{[(1+\varepsilon)(\overline{\delta}_\lambda(x_0)+\varepsilon)(\overline{\ell}+\varepsilon)+\varepsilon]/s\log\beta}\,. \tag{87}$$

From (86) and (87) we get (83).

□

Remark 6.5. If $X \subset \mathbf{R}$ and λ is Lebesgue measure, then Theorem 6.1 holds also for all approximable point $x_0 \in X_0^+$. In fact, in this case we can not assure that closure$(P(k_1, x_0)) \subset P(0, x_0)$. However it is true that closure$(P(k_1, x_0)) \subset P(0, x_0) \cup \{x_0\}$ and from this fact we can conclude easily that

$$\mathcal{C}_\varepsilon := \bigcap_{n=0}^{\infty} \bigcup_{J\in\mathcal{J}_n} J \subset \bigcap_{n=0}^{\infty} \bigcup_{J\in\mathcal{J}_n} \mathrm{closure}\,(J) \subset \mathcal{C}_\varepsilon \cup S\,,$$

where S is a countable set. Hence, $\mathcal{C}_\varepsilon$ and $\bigcap_{n=0}^{\infty}\bigcup_{J\in\mathcal{J}_n} \mathrm{closure}\,(J)$ have the same Hausdorff dimensions. Also, since $\lambda(J) = \lambda(\mathrm{closure}\,(J))$ the proof of Theorem 6.1 allows to estimate the Hausdorff dimensions of $\bigcap_{n=0}^{\infty}\bigcup_{J\in\mathcal{J}_n} \mathrm{closure}\,(J)$.

The next result follows from the proof of Theorem 6.1. In this case the sequence of radii is constant and therefore we are estimating the set of points returning periodically to a neighbourhood of the given point x_0. The proof is much more simple because the constructed Cantor-like sets have a more regular pattern.

Corollary 6.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Let $r > 0$ and let P be a block of $\mathcal{P}_{N_0}$. Then, given $\varepsilon > 0$, for all point $x_0 \in X_0$, there exist k depending on x_0 and r, and $\widetilde{N}$ depending on P, x_0, r and ε such that for all $N \ge \widetilde{N}$ the grid Hausdorff dimensions of the set $\mathcal{R}(P, x_0, r, N)$ of points $x \in P \cap X_0$ such that*

$$d(T^{d_j}(x), x_0) < r \quad \textit{for} \quad d_j = N_0 + k + (j-1)(N+k) \quad \textit{for } j = 1, 2, \ldots$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{R}(P, x_0, r, N)) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\mathcal{R}(P, x_0, r, N)) \ge 1 - \varepsilon - C_1/N\,.$$

where C_1 is an absolute constant. Moreover for all $x_0 \in X_0$ we have

1. *If the grid $\mathbf{\Pi}$ is λ-regular then,*

$$\mathrm{Dim}_\lambda(\mathcal{R}(U, x_0, r, N)) = \mathrm{Dim}_\mu(\mathcal{R}(U, x_0, r, N)) \ge 1 - \varepsilon - C_2/N.$$

2. *If λ is a doubling measure verifying that $\lambda(B(x, r)) \le C\,r^s$ for all ball $B(x, r)$, then*

$$\mathrm{Dim}_\lambda(\mathcal{R}(P, x_0, r, N)) = \mathrm{Dim}_\mu(\mathcal{R}(P, x_0, r, N)) \ge 1 - \frac{\log C_3}{(N+k)s\log\beta}\,,$$

   with $C_3 \asymp 1/\lambda(P(k, x_0))$.

Proof. We have now $\bar{\ell} = 0$ and we can do the same construction that in Theorem 6.1 with $N_j = N$ and $k_j = k$ for all $j \geq 1$. The result for $\mathrm{Dim}_{\mathbf{\Pi},\lambda}$ follows from Corollary 2.1 by taking $\alpha_j = e^{-Na}$, $\beta_j = e^{-Nb}$ with $a = h_\mu + 2\varepsilon$, $b = h_\mu - 2\varepsilon$ and γ_j a constant. Part 1 is a consequence of Proposition 2.1. The proof of Part 2 is similar to the corresponding one in the proof of Theorem 6.1. Now instead of (84) we have that $\nu(P(m,x)) \leq C\, C_3^n \lambda(P(m,x))$. $\square$

Lemma 6.2. *Let $\{A_k\}$ be a decreasing sequence of Borel sets in X such that* $\mathrm{Dim}_{\mathbf{\Pi},\lambda}(A_k) \geq \beta > 0$. *Then, $Dim_{\mathbf{\Pi},\lambda}(\cap_k A_k) \geq \beta$.*

Proof. If $0 < \alpha < \beta$, then $\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A_k) = \infty$ for all k and therefore $\mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(\cap_k A_k) = \lim_{k\to\infty} \mathcal{H}^\alpha_{\mathbf{\Pi},\lambda}(A_k) = \infty$. It follows that $\mathrm{Dim}_\lambda(\cap_k A_k) \geq \alpha$. The result follows by letting $\alpha \to \beta$. $\square$

Remark 6.6. The lemma also holds (with the same proof) for the λ-Hausdorff dimension.

Corollary 6.2. *Under the hypotheses of Theorem* 6.1 *we have that*

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda} \left\{ x \in U : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0 \right\} \geq \frac{h_\mu}{h_\mu + \bar{\delta}_\lambda(x_0)\, \bar{\ell}}\,.$$

Moreover, if the grid $\mathbf{\Pi}$ is λ-regular, then

$$\mathrm{Dim}_\lambda \left\{ x \in U : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0 \right\} \geq \frac{h_\mu}{h_\mu + \bar{\delta}_\lambda(x_0)\, \bar{\ell}} \left(1 - \frac{\overline{\tau}_\lambda(x_0)\bar{\delta}_\lambda(x_0)\ell^2}{h_\mu^2 \log \beta}\right).$$

Proof. Notice that if $x \in U$ verifies

$$d(T^n(x), x_0) \leq r_n\,, \text{ for infinitely many } n \quad \Longrightarrow \quad \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq 1$$

and from Theorem 6.1 we obtain that

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda} \left\{ x \in U : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq 1 \right\} \geq \frac{h_\mu}{h_\mu + \bar{\delta}_\lambda(x_0)\, \bar{\ell}}\,.$$

By applying this last result to the sequence $\{r_n/m\}_{n=1}^\infty$ for any $m \in \mathbf{N}$, we get that

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda} \left\{ x \in U : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq \frac{1}{m} \right\} \geq \frac{h_\mu}{h_\mu + \bar{\delta}_\lambda(x_0)\, \bar{\ell}}\,,$$

and since

$$\left\{ x \in X : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} = 0 \right\} = \bigcap_{m=1}^\infty \left\{ x \in X : \liminf_{n\to\infty} \frac{d(T^n(x), x_0)}{r_n} \leq \frac{1}{m} \right\}.$$

the lower bound in the statement follows from the above lemma. The proof of the second statement is similar. $\square$

As in the measure section we are also interested in the size of the set

$$\widetilde{\mathcal{W}}(x_0, \{t_n\}) = \{x \in X_0 : \; T^k(x) \in P(t_k, x_0) \text{ for infinitely many } k\}$$

with $\{t_k\}$ an increasing sequence of positive integers and $x_0 \in X_0^+$. We recall that if $x_0 = [\, i_0\, i_1 \ldots]$, then $\widetilde{\mathcal{W}}(x_0, \{t_n\})$ is the set of points $x = [\, m_0\, m_1 \ldots] \in X_0$ such that

$$m_k = i_0, m_{k+1} = i_i, \; \ldots \; , m_{k+t_k} = i_{t_k}$$

for infinitely many k.

Theorem 6.2. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let $\{t_n\}$ be a non decreasing sequence of positive integers and let U be an open set in X with $\mu(U) > 0$. Let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Then, for all approximable point $x_0 \in X_0$, the grid Hausdorff dimensions of the set*

$$\widetilde{\mathcal{W}}(U, x_0, \{t_n\}) = \{x \in U \cap X_0 : \ T^k(x) \in P(t_k, x_0) \text{ for infinitely many } k\} ,$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \geq \frac{h_\mu}{h_\mu + \overline{L}(x_0)} ,$$

where $\overline{L}(x_0) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda(P(t_n,x_0))}$ *and h_μ is the entropy of T with respect to μ.*

Moreover, for all approximable point $x_0 \in X_0$, the Hausdorff dimension of the set $\widetilde{\mathcal{W}}(U, x_0, \{t_n\})$ verify:

1. *If the grid $\mathbf{\Pi}$ is λ-regular, then*

$$\mathrm{Dim}_\lambda(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_\mu(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \geq \frac{h_\mu}{h_\mu + \overline{L}(x_0)} \Big(1 - \frac{\overline{\tau}_\lambda(x_0)\, \overline{w}\, \overline{L}(x_0)}{h_\mu^2}\Big) ,$$

   where $\overline{w} = \limsup_{n\to\infty} \frac{t_n}{n}$.

2. *If λ is a doubling measure verifying that $\lambda(B(x, r)) \leq C\, r^s$ for all ball $B(x, r)$, then*

$$\mathrm{Dim}_\lambda(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_\mu(\mathcal{W}(U, x_0, \{t_n\})) \geq 1 - \frac{\overline{L}(x_0)\overline{w}}{s \log \beta} .$$

Remark 6.7. We recall that from Remark 6.1 and Lemma 4.4 we know that the set of approximable points such that $\overline{\tau}_\lambda(x_0) = 0$ has full λ-measure.

As in the case of radii, we have the following consequence of the proof of Theorem 6.2 when we take the sequence $\{t_n := t\}$ constant. We are estimating the set of points in whose code appear periodically the first t digits of the code of the point x_0. The proof is similar.

Corollary 6.3. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$ where μ is the ACIPM associated to the system. Let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Let $t \in \mathbf{N}$ and let P be an block of $\mathcal{P}_{N_0}$. Then, given $\varepsilon > 0$ for all point $x_0 \in X_0$ there exist k depending on x_0 and t, and $\widetilde{N}$ depending on P, x_0 ε and t, such that for all $N \geq \widetilde{N}$ the grid Hausdorff dimensions of the set $\widetilde{\mathcal{R}}(P, x_0, r, N)$ of points $x = [\ m_0\ m_1 \ldots\ ] \in P \cap X_0$ such that for $j = 1, 2, \ldots$*

$$m_{d_j} = i_0,\, m_{d_j+1} = i_1,\ \ldots\ , m_{d_j+t} = i_t \quad \text{with} \quad d_j = N_0 + k + (j-1)(N + k)$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\mathcal{R}(P, x_0, r, N)) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\mathcal{R}(P, x_0, r, N)) \geq 1 - \varepsilon - C_1/N ,$$

where C_1 is an absolute constant. Moreover for all $x_0 \in X_0$ we have

1. *If the grid $\mathbf{\Pi}$ is λ-regular then,*

$$\mathrm{Dim}_\lambda(\mathcal{R}(U, x_0, r, N)) = \mathrm{Dim}_\mu(\mathcal{R}(U, x_0, r, N)) \geq 1 - \varepsilon - C_2/N.$$

2. *If λ is a doubling measure verifying that $\lambda(B(x, r)) \leq C\, r^s$ for all ball $B(x, r)$, then*

$$\mathrm{Dim}_\lambda(\mathcal{R}(P, x_0, r, N)) = \mathrm{Dim}_\mu(\mathcal{R}(P, x_0, r, N)) \geq 1 - \frac{\log C_3}{(N + k)s \log \beta} ,$$

   with $C_3 \asymp 1/\lambda(P(k, x_0))$.

Proof of Theorem 6.2. We may assume that $\overline{L}(x_0)$, $\overline{\tau}_\lambda(x_0)$ and $\overline{w}$ are all finite, since otherwise our Hausdorff dimension estimates are obvious. Now, the proof is similar to the proof of Theorem 6.1. For each $\varepsilon > 0$ we construct a Cantor-like set $\mathcal{C}_\varepsilon \subset \widetilde{\mathcal{W}}(U, x_0, \{t_n\})$. Recall that in the proof of Theorem 6.1 we defined an increasing sequence $\mathcal{D}$ of allowed indexes. Here, we define $\mathcal{D}$ in the following way: Let $\mathcal{I}(x_0) = \{p_i\}$ denote the sequence associated to x_0 given by Definition 6.1. For each $p_k \in \mathcal{I}(x_0)$ let $n(k)$ denote the greatest natural number such that $t_{n(k)} \leq p_k$. We denote by $\mathcal{D}$ the set of these allowed indexes. We will write $\mathcal{D} = \{d_i\}$ with $d_i < d_{i+1}$.

With this new definition of $\mathcal{D}$ we have that if $d \in \mathcal{D}$, then there exists $k(d) \in \mathcal{I}(x_0)$ such that

$$t_d \le k(d) < t_{d+1} \qquad \text{and} \qquad P(k(d), x_0) \subset P(t_d, x_0)\,.$$

These two properties substitute to (42) and (43). For all d large enough we have that

$$\frac{k(d)}{d} \le (1+\varepsilon)\overline{w}\,.$$

This inequality substitute to (44). With the above considerations and proceeding as in the proof of Theorem 6.1 we construct the families $\widetilde{\mathcal{J}}_j$, $\mathcal{J}_j$ and the numbers N_j and $k_j \in \mathcal{I}(x_0)$ with the properties (b), (c1), (c3), (d) and (e). The corresponding properties (a) and (c2) are now the following ones:

(a) For all point x in $J_j \in \mathcal{J}_j$

$$T^{d_j}(x) \in P(t_{d_j}, x_0)\,.$$

(c2) For all $J_j \in \mathcal{J}_j$ there exist a unique $\widetilde{J}_j \in \widetilde{\mathcal{J}}_j$ so that closure$\,(J_j) \subset \widetilde{J}_j$ and

$$\frac{\lambda(J_j)}{\lambda(\widetilde{J}_j)} \asymp \lambda(P(k_j, x_0)) \qquad \text{and} \qquad \frac{\lambda(J_j)}{\lambda(\widetilde{J}_j)} \ge e^{-N_j[(1+\varepsilon)(\overline{L}(x_0)+\varepsilon)+\varepsilon]}\,.$$

The rest of the proof is similar. □

Remark 6.8. Using the same argument that in Remark 6.5 we get that if $X \subset \mathbf{R}$ and λ is Lebesgue measure, then Theorem 6.2 holds also for all approximable point $x_0 \in X_0^+$.

For points $x_0 \in X_1$ we have the following version of the above theorem.

Theorem 6.3. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system with finite entropy $H_\mu(\mathcal{P}_0)$ with respect to the partition $\mathcal{P}_0$, where μ is the ACIPM associated to the system. Let $\{t_n\}$ be a non decreasing sequence of positive integers and let U be an open set in X with $\mu(U) > 0$. Let us consider the grid $\mathbf{\Pi} = \{\mathcal{P}_n\}$. Then, for all approximable point $x_0 \in X_1$, and therefore for λ-almost point $x_0 \in X$, the grid Hausdorff dimensions of the set*

$$\widetilde{\mathcal{W}}(U, x_0, \{t_n\}) = \{x \in U \cap X_0 :\ T^k(x) \in P(t_k, x_0) \textit{ for infinitely many } k\}\,,$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \ge \frac{1}{1+\overline{w}}\,,$$

where $\overline{w} = \limsup_{n\to\infty} \frac{t_n}{n}$. Moreover, if the grid $\mathbf{\Pi}$ is λ-regular, then also

$$\mathrm{Dim}_{\lambda}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_{\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \ge \frac{1}{1+\overline{w}}\,.$$

Proof. We may assume that $\overline{w} < \infty$ since otherwise our estimations are trivial. The proof is similar to the proof of Theorem 6.2 but using that for d large

$$\lambda(P(t_d, x_0)) \ge e^{-t_d(h_\mu+\varepsilon)} \ge e^{-d(\overline{w}+\varepsilon)(h_\mu+\varepsilon)}\,.$$

□

6.2 Upper bounds of the dimension

We will prove now some upper bounds for the λ-grid Hausdorff dimension of $\mathcal{W}(U, x_0, \{r_n\})$ and $\widetilde{\mathcal{W}}(U, x_0, \{t_n\})$ in the case that the partition $\mathcal{P}_0$ is finite.

Proposition 6.1. *Let $(X, d, \mathcal{A}, \lambda, T)$ be an expanding system such that the partition $\mathcal{P}_0$ is finite. Let μ be the ACIPM associated to the system. Let $\{t_n\}$ be a non decreasing sequence of positive integers and U be an open set in X with $\mu(U) > 0$.*

Then, if $x_0 \in X_0^+$, we have that the grid Hausdorff dimensions of the set

$$\widetilde{\mathcal{W}}(U, x_0, \{t_n\}) = \{x \in U \cap X_0 :\ T^k(x) \in P(t_k, x_0) \textit{ for infinitely many } k\}\,,$$

verify

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \le \min\left\{1, \frac{\log D}{h_\mu + \underline{L}(x_0)}\right\}\,.$$

where $\underline{L}(x_0) = \liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{\lambda(P(t_n,x_0))}$, $h_\mu = h_\mu(T)$ *is the entropy of* T *with respect to the measure* μ *and* D *is the cardinality of* $\mathcal{P}_0$. *Moreover, if* $x_0 \in X_1$, *then*

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) = \mathrm{Dim}_{\mathbf{\Pi},\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \le \min\left\{1, \frac{\log D}{(1+\underline{w})h_\mu}\right\},$$

where $\underline{w} = \liminf_{n\to\infty} \frac{1}{n}\, t_n$.

Proof. We define the collections

$$\mathcal{F}_n = \{T^{-n}(P(t_n, x_0)) \cap Q : \ Q \in \mathcal{P}_n\}.$$

Then the collection

$$\mathcal{G}_N = \bigcup_{n=N}^{\infty} \mathcal{F}_n .$$

covers the set $\widetilde{\mathcal{W}}(U, x_0, \{t_n\})$ for all $N \in \mathbf{N}$. Using Proposition 4.3 we get that

$$\sum_{k=N}^{\infty} \sum_{F\in\mathcal{F}_k} \mu(F)^\tau \le C \sum_{k=N}^{\infty} \mu(P(t_k, x_0))^\tau \sum_{Q\in\mathcal{P}_k} \mu(Q)^\tau . \tag{88}$$

Let us consider now the following two subcollections of $\mathcal{P}_k$:

$$\mathcal{P}_{k,\mathrm{small}} = \{Q \in \mathcal{P}_k : \ \mu(Q) \le e^{-kh_\mu}\}, \qquad \mathcal{P}_{k,\mathrm{big}} = \{Q \in \mathcal{P}_k : \ \mu(Q) > e^{-kh_\mu}\}.$$

Then,

$$\sum_{Q\in\mathcal{P}_{k,\mathrm{small}}} \mu(Q)^\tau \le D^k e^{-k\tau h_\mu}$$

and

$$\sum_{Q\in\mathcal{P}_{k,\mathrm{big}}} \mu(Q)^\tau = \sum_{Q\in\mathcal{P}_{k,\mathrm{big}}} \frac{1}{\mu(Q)^{1-\tau}}\, \mu(Q) \le e^{kh_\mu(1-\tau)} .$$

Since $h_\mu \le H_\mu(\mathcal{P}_0) \le \log D$ we have that

$$e^{kh_\mu(1-\tau)} \le D^k e^{-k\tau h_\mu}$$

and therefore using (88) we obtain that

$$\sum_{k=N}^{\infty} \sum_{F\in\mathcal{F}_k} \mu(F)^\tau \le 2C \sum_{k=N}^{\infty} D^k e^{-k\tau h_\mu} \mu(P(t_k, x_0))^\tau$$

By part (iii) of Theorem E we know that $\mu(P(t_k, x_0)) \asymp \lambda(P(t_k, x_0))$. Then for t_k large enough

$$\mu(P(t_k, x_0)) \asymp \lambda(P(t_k, x_0)) \le e^{-k(\underline{L}-\varepsilon)} .$$

Hence

$$\sum_{G\in\mathcal{G}_N} \mu(G)^\tau \to 0 \qquad \text{when} \quad N \to \infty$$

for

$$\tau > \frac{\log D}{h_\mu + \underline{L}(x_0) - \varepsilon} .$$

For these τ's the τ-dimensional μ-grid Hausdorff measure of $\widetilde{\mathcal{W}}(U, x_0, \{t_n\})$ is zero and therefore

$$\mathrm{Dim}_{\mathbf{\Pi},\mu}(\widetilde{\mathcal{W}}(U, x_0, \{t_n\})) \le \frac{\log D}{h_\mu + \underline{L}(x_0) - \varepsilon} .$$

The result follows by taking ε tending to zero and using Lemma 6.1. Finally, if $x_0 \in X_1$, we have that $\underline{L}(x_0) = h_\mu\, \underline{w}$. $\square$

Remark 6.9. Even in the case that the partition $\mathcal{P}_0$ is not finite a slight modification of the above proof shows that

$$\mathrm{Dim}_{\mathbf{\Pi},\lambda}(\widetilde{\mathcal{W}}(U,x_0,\{t_n\})\cap X_1)\le \frac{h_\mu}{h_\mu+\underline{L}(x_0)}\,. \tag{89}$$

To see this, notice that if we define for any $\varepsilon>0$ the subcollections:

$$\mathcal{P}_{n,\varepsilon,\mathrm{big}}=\{Q\in\mathcal{P}_n:\ \mu(Q)>e^{-n(h_\mu+\varepsilon)}\}\,.$$

then the set $\widetilde{\mathcal{W}}(U,x_0,\{t_n\})\cap X_1$ can be covered by the collections

$$\mathcal{G}_N^\varepsilon=\bigcup_{n=N}^{\infty}\mathcal{F}^\varepsilon_{n,\mathrm{big}}\,.$$

where

$$\mathcal{F}^\varepsilon_{n.\mathrm{big}}=\{T^{-n}(P(t_n,x_0))\cap Q:\ Q\in\mathcal{P}_{n,\varepsilon,\mathrm{big}}\}\,.$$

The proof of (89) follows now easily.

Proposition 6.2. *Let $(X,d,\mathcal{A},\lambda,T)$ be an expanding system with $X\subset\mathbf{R}$ and such that the partition $\mathcal{P}_0$ is finite. Let μ be the ACIPM associated to the system. Let $\{r_n\}$ be a non increasing sequence of positive numbers and U be an open set in X with $\mu(U)>0$.*

Then, for $x_0\in X_0$, the Hausdorff dimensions of the set

$$\mathcal{W}(U,x_0,\{r_n\})=\{x\in U\cap X_0:\ d(T^n(x),x_0)\le r_n \text{ for infinitely many } n\}$$

verify

$$\mathrm{Dim}_\lambda(\mathcal{W}(U,x_0,\{r_n\}))=\mathrm{Dim}_\mu(\mathcal{W}(U,x_0,\{r_n\}))\le\min\left\{1,\frac{\log D}{h_\mu+\underline{\delta}_\lambda(x_0)\underline{\ell}}\right\}, \tag{90}$$

where D is the cardinality of $\mathcal{P}_0$, $h_\mu=h_\mu(T)$ is the entropy of T with respect to μ and

$$\underline{\ell}=\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{r_n}$$

Proof. We define the collections

$$\mathcal{F}_n=\{T^{-n}(B(x_0,r_n))\cap Q:\ Q\in\mathcal{P}_n\}\,.$$

Notice that for n large enough $B(x_0,r_n)\subseteq P(0,x_0)$ and then $T^{-n}(B(x_0,r_n))\cap Q$ is an interval for any $Q\in\mathcal{P}_n$. Therefore, the collection of intervals

$$\mathcal{G}_N=\bigcup_{n=N}^{\infty}\mathcal{F}_n$$

covers the set $\mathcal{W}(U,x_0,\{r_n\})$ for all $N\in\mathbf{N}$ large enough. Using Proposition 4.3 we get that

$$\sum_{k=N}^{\infty}\sum_{F\in\mathcal{F}_k}\mu(F)^\tau\le C\sum_{k=N}^{\infty}\mu(B(x_0,r_k))^\tau\sum_{Q\in\mathcal{P}_k}\mu(Q)^\tau\,.$$

By estimating $\sum_{Q\in\mathcal{P}_k}\mu(Q)^\tau$ as in the proof of Proposition 6.1 we get

$$\sum_{k=N}^{\infty}\sum_{F\in\mathcal{F}_k}\mu(F)^\tau\le 2C\sum_{k=N}^{\infty}D^ke^{-k\tau h_\mu}\mu(B(x_0,r_k))^\tau$$

For r_k small enough we have that $B(x_0,r_k)\subset P(0,x_0)$ and then $\mu(B(x_0,r_k))\asymp\lambda(B(x_0,r_k))$ by part (iii) of Theorem E. Hence, from the definition of $\underline{\delta}_\lambda(x_0)$ and Lemma 3.1, we conclude that given $\varepsilon>0$,

$$\sum_{k=N}^{\infty}\sum_{F\in\mathcal{F}_k}\mu(F)^\tau\le C\sum_{k=N}^{\infty}r_k^{(\underline{\delta}_\lambda(x_0)-\varepsilon)\tau}D^ke^{-k\tau h_\mu}\longrightarrow 0$$

as $N\to\infty$, if

$$\tau>\frac{\log D}{h_\mu+(\underline{\delta}_\lambda(x_0)-\varepsilon)(\underline{\ell}-\varepsilon)}\,.$$

For these τ's the τ-dimensional μ-Hausdorff measure of $\mathcal{W}(U,x_0,\{r_n\})$ is zero and therefore

$$\mathrm{Dim}_\mu(\mathcal{W}(U,x_0,\{r_n\}))\le\frac{\log D}{h_\mu+(\underline{\delta}_\lambda(x_0)-\varepsilon)(\underline{\ell}-\varepsilon)}\,.$$

The result follows by taking ε tending to zero and using Lemma 6.1. $\square$

7 Applications

7.1 Markov transformations

Let λ be Lebesgue measure in $[0,1]$. A map $f:[0,1]\longrightarrow[0,1]$ is a *Markov transformation* if there exists a family $\mathcal{P}_0=\{P_j\}$ of disjoint open intervals in $[0,1]$ such that

(a) $\lambda([0,1]\setminus\cup_j P_j)=0$.

(b) For each j, there exists a set K of indices such that $f(P_j)=\cup_{k\in K}P_k$ (mod 0).

(c) f is derivable in $\cup_j P_j$ and there exists $\sigma>0$ such that $|f'(x)|\geq\sigma$ for all $x\in\cup_j P_j$.

(d) There exists $\gamma>1$ and a non zero natural number n_0 such that if $f^m(x)\in\cup_j P_j$ for all $0\leq m\leq n_0-1$, then $|(f^{n_0})'(x)|\geq\gamma$.

(e) There exists a non zero natural number m such that $\lambda(f^{-m}(P_i)\cap P_j)>0$ for all i,j.

(f) There exist constants $C>0$ and $0<\alpha\leq 1$ such that, for all $x,y\in P_j$,

$$\Big|\frac{f'(x)}{f'(y)}-1\Big|\leq C|f(x)-f(y)|^{\alpha}\,.$$

Markov transformations are expanding maps with parameters α and $\beta=\gamma^{1/n_0}$, see [30], p.171, and therefore, by Theorem E, there exists a unique f-invariant probability measure μ in $[0,1]$ which is absolutely continuous with respect to Lebesgue measure and satisfies properties (i)-(v) in Theorem E. As a consequence of our results we obtain

Theorem 7.1. *Let $f:[0,1]\longrightarrow[0,1]$ be a Markov transformation and $\{r_n\}$ be a non increasing sequence of positive numbers. Then,*

(1) *If $\sum_n r_n^{1+\varepsilon}=\infty$ for some $\varepsilon>0$, then for almost all $x_0\in[0,1]$ we have that*

$$\liminf_{n\to\infty}\frac{|f^n(x)-x_0|}{r_n}=0\,,\qquad \textit{for almost all } x\in[0,1].$$

(2) *If $\sum_n r_n<\infty$, then for all $x_0\in\cup_j P_j$ we have that*

$$\liminf_{n\to\infty}\frac{|f^n(x)-x_0|}{r_n}=\infty\,,\qquad \textit{for almost all } x\in[0,1].$$

(3) *If $H_\mu(\mathcal{P}_0)<\infty$, then, for almost all $x_0\in[0,1]$, we have that*

$$\operatorname{Dim}\left\{x\in[0,1]:\ \liminf_{n\to\infty}\frac{|f^n(x)-x_0|}{r_n}=0\right\}\geq\frac{h_\mu}{h_\mu+\overline{\ell}}\,,$$

where $\overline{\ell}=\limsup\frac{1}{n}\log\frac{1}{r_n}$, $h_\mu=\int_0^1\log|f'(x)|\,d\mu(x)$ and Dim *denotes Hausdorff dimension*.

If the partition $\mathcal{P}_0$ is finite, then all the statements hold for all $x_0\in[0,1]$.

Let us observe that if $\sum_n r_n=\infty$ the theorem does not tell us what is the measure of the set where

$$\liminf_{n\to\infty}\frac{|f^n(x)-x_0|}{r_n}=0$$

but by part (3) we know that this set is big since has positive Hausdorff dimension.

Remark 7.1. For sake of simplicity we have stated the above theorem for almost all point x_0. However, the results in the previous sections give more information if we choose an specific x_0, see Remarks 5.3 and 6.1-6.4.

Proof of Theorem 7.1. Part (1) follows from Lemma 4.4 and Corollary 5.4. Part (2) is a consequence of Corollary 5.2. Finally, part (3) follows from Remark 6.1, Remark 2.4, Lemma 4.4 and Corollary 6.2. Finally, to get the result when the partition $\mathcal{P}_0$ is finite, we use additionally that, in this case, $X_0^+=[0,1]$, $\overline{\tau}_\lambda(x_0)=0$ for all $x_0\in[0,1]$ and Remarks 5.1, 6.2 and 6.5. $\square$

Recall that, as we saw in Section 4.1, given an expanding map we have a code for almost all point x_0, and more precisely for all $x_0\in X_0^+$. The following result summarizes our results about coding for Markov transformations.

Theorem 7.2. *Let $f : [0,1] \longrightarrow [0,1]$ be a Markov transformation and $\{t_n\}$ be a non decreasing sequence of natural numbers. Given a point $x_0 = [\,i_0, i_1, \ldots] \in X_0^+$, let $\widetilde{\mathcal{W}}(x_0, \{t_n\})$ be the set of points $x = [\,m_0, m_1, \ldots] \in X_0$ such that*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \ldots\,,\ m_{n+t_n} = i_{t_n}\,, \qquad \text{for infinitely many } n.$$

Then,

(1) *If $\sum_n \lambda(P(t_n, x_0)) = \infty$, then $\lambda(\widetilde{W}(x_0, \{t_n\}))) = 1$. Moreover, if the partition $\mathcal{P}_0$ is finite or if $f(P) = [0,1]$ (mod 0) for all $P \in \mathcal{P}_0$, then we have the following quantitative version:*

$$\lim_{n\to\infty} \frac{\#\{i \le n :\ f^i(x) \in P(t_i, x_0)\}}{\sum_{j=1}^{n} \mu(P(t_j, x_0))} = 1\,, \qquad \text{for } \lambda\text{-almost every } x.$$

(2) *If $\sum_n \lambda(P(t_n, x_0)) < \infty$, then $\lambda(\widetilde{W}(x_0, \{t_n\}))) = 0$.*

(3) *If $H_\mu(\mathcal{P}_0) < \infty$, then, for almost all $x_0 \in X$, we have that*

$$\mathrm{Dim}(\widetilde{\mathcal{W}}(x_0, \{t_n\}) \ge \frac{1}{1+\overline{w}}\,.$$

where $\overline{w} = \limsup_{n\to\infty} \frac{t_n}{n}$ and Dim denotes Hausdorff dimension.

Remark 7.2. Even though part (3) is stated for almost every x_0 a more precise result for an specific x_0 follows from Theorem 6.2 and Remark 6.8. Recall also that any grid contained in $\mathbf{R}$ is regular.

Proof of Theorem 7.2. Part (1) and (2) follow from Theorem 5.1 and Proposition 5.1, respectively. Part (3) is a consequence of Lemma 4.4, Remark 6.1, Remark 2.4 and Theorem 6.3. ☐

7.1.1 Bernoulli shifts and subshifts of finite type

Given a natural number D let Σ denote the space of all infinite sequences $\{(i_0\,,\,i_1\,,\,\ldots)\}$ with $i_n \in \{0, 1, \ldots, D-1\}$ endowed with the product topology. The left shift $\sigma : \Sigma \longrightarrow \Sigma$ is the continuous map defined by

$$\sigma(i_0\,,\,i_1\,,\,\ldots) = (i_1\,,\,i_2\,,\,\ldots)\,.$$

For every positive numbers $p_0, p_1, \ldots, p_{D-1}$ verifying $\sum_{i=0}^{D-1} p_i = 1$ we define the function

$$\nu(C^{j_0, j_1, \ldots, j_t}_{i_0, i_1, \ldots, i_t}) = p_{i_0} p_{i_1} \cdots p_{i_t}\,,$$

where $C^{j_0, j_1, \ldots, j_t}_{i_0, i_1, \ldots, i_t}$ is the cylinder

$$C^{j_0, j_1, \ldots, j_t}_{i_0, i_1, \ldots, i_t} = \{(k_0\,, k_1\,, \ldots) \in \Sigma :\ k_{j_s} = i_s \text{ for all } s = 0, 1, \ldots, t\}\,.$$

It is well known that we can extend the set function ν to a probability measure defined on the σ-algebra of the Borel sets of Σ. The space (Σ, σ, ν) is called a (one-sided) *Bernoulli shift.*

We can generalize the full shift space (Σ, σ, ν) by considering the set Σ_A defined by

$$\Sigma_A = \{(i_0\,,\,i_1\,,\,\ldots) \in \Sigma :\ a_{i_k, i_{k+1}} = 1 \text{ for all } k = 0, 1, \ldots\}\,,$$

where $A = (a_{i,j})$ is a $D \times D$ matrix with entries $a_{i,j} = 0$ or 1. The matrix A is known as a *transition matrix.* Let us consider now a new $D \times D$ matrix $M = (p_{i,j})$ such that $p_{i,j} = 0$ if $a_{i,j} = 0$, and

$$(1) \qquad \sum_{j=0}^{D-1} p_{i,j} = 1\,, \quad \text{for every } i = 0, 1, \ldots, D-1.$$

$$(2) \qquad \sum_{i=0}^{D-1} p_i p_{i,j} = p_j\,, \quad \text{for every } j = 0, 1, \ldots, D-1.$$

The numbers $p_{i,j}$ are called the *transition probabilities* associated to the transition matrix A and the matrix M is called a *stochastic matrix.* Observe that the probability vector $(p_0, \ldots, p_{D-1})$ is an eigenvector of the matrix M.

We introduce now a probability measure ν on all Borel subsets of Σ_A by extending the set function defined by

$$\nu(C^{n,\dots,n+t}_{i_0,i_1,\dots,i_t}) = p_{i_0} p_{i_0,i_1} \cdots p_{i_{t-1},i_t} \, .$$

The space (Σ_A, σ, ν) is called a (one-sided) *subshift of finite type* or a (one-sided) *Markov chain.*

We will explain now how to associate to (one-sided) Bernoulli shifts or (one-sided) subshift of finite type a Markov transformation:

Let (Σ, σ, ν) be a (one-sided) Bernoulli shift and let λ denote the Lebesgue measure in $[0,1]$. Consider a partition $\{P_0, \dots, P_{D-1}\}$ of $[0,1]$ in D consecutive open intervals such that $\lambda(P_j) = p_j$ for $j = 0, 1, \dots, D-1$. We define now a function $f : [0,1] \longrightarrow [0,1]$ by letting f to be linear and bijective from each I_j onto $(0,1)$, i.e.

$$f(x) = \frac{1}{p_j} \left(x - \sum_{k=0}^{j-1} p_k \right) , \qquad \text{if } x \in P_j \, ,$$

and f equal to zero on the boundaries of the intervals P_j. It is easy to check that f is a Markov transformation and therefore an expanding map.

Define now a mapping $\pi : \Sigma \longrightarrow [0,1]$ by

$$\pi((i_0, i_1, \dots)) = \bigcap_{n=0}^{\infty} \text{closure} \left(f^{-n}(P_{i_n}) \right) .$$

Then it is not difficult to see that π is continuous and $f \circ \pi = \pi \circ \sigma$.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ [0,1] & \xrightarrow{f} & [0,1] \end{array}$$

Notice that the space of codes associated through f to the points in X_0 (as we explained in Section 4.1) is precisely the set Σ_0 of all sequences $(i_0, i_1, \dots)$ such that there is not $k_0 \in \mathbf{N}$ such that $i_j = 0$ for all $j \geq k_0$ or $i_j = D-1$ for all $j \geq k_0$, and that π is bijective from this set onto $X_0 = [0,1] \setminus \cup_{n=0}^{\infty} f^{-n}(\cup_i \partial P_i)$.

It is also easy to check that the image measure of the product measure ν under π is precisely the Lebesgue measure in $[0,1]$ and that f preserves Lebesgue measure. Therefore the dynamical systems (Σ, σ, ν) and $([0,1], f, \lambda)$ are isomorphic. We have also that the Hausdorff dimension of a Borel subset $B \subset [0,1]$ coincides que the ν-Hausdorff dimension of $\pi^{-1}(B)$.

The subshifts of finite type whose stochastic matrix M is transitive (i.e. there exists $n_0 > 0$ such that all entries of M^{n_0} are positive) can be also thought as Markov transformations: Let (Σ_A, σ, ν) be a subshift of finite type with respect to the stochastic matrix $M = (p_{i,j})$ and the probability vector $(p_0, p_1, \dots, p_{D-1})$. Consider as before a partition $\{P_0, P_1, \dots, P_{D-1}\}$ of $[0,1]$ in D consecutive open intervals such that $\lambda(P_i) = p_i$ for $i = 0, 1, \dots, D-1$. We divide now each interval P_i into D consecutive open intervals $P_{i,j}$ such that $\lambda(P_{i,j}) = p_i p_{i,j}$ for $j = 0, 1, \dots, D-1$. If $p_{i,j} = 0$ we take $P_{i,j} = \emptyset$. Notice that by property (1) of stochastic matrices we have that $\lambda(P_i) = \sum_j \lambda(P_{i,j})$.

We define now a function $f : [0,1] \longrightarrow [0,1]$ in the following way: for each $P_{i,j} \neq \emptyset$,

$$f(x) = \frac{1}{p_{j,i}} \left(x - \sum_{k=0}^{j-1} p_k p_{k,i} - \sum_{\ell=0}^{i-1} p_\ell \right) + \sum_{k=0}^{j-1} p_k \, , \qquad \text{if } x \in P_{i,j} \, .$$

We define also f on the points beloging to $P_i \cap \partial P_{i,j}$ in such a way that f is continuous in that points. Finally we define f to be zero on the boundaries of the intervals P_i.

When the stochastic matrix M verifies that there exists n_0 such that all entries of M^{n_0} are positive, it is easy to check that f is a Markov transformation and therefore an expanding map with respect to the partition $\mathcal{P}_0 = \{P_i : \ i = 0, 1, \dots, D-1\}$. The condition on M it is necessary only to assure property (e) of Markov transformations, see [30], Lemma 12.2. Notice also that the condition (2) in the definition of stochastic matrices means that f preserves Lebesgue measure.

As in the case of Bernoulli shifts, the dynamical systems (Σ_A, σ, ν) and $([0,1], f, \lambda)$ are isomorphic and also the Hausdorff dimension of a Borel subset $B \subset [0,1]$ coincides que the ν-Hausdorff dimension of $\pi^{-1}(B)$.

We obtain the following result:

Corollary 7.1. *Let (Σ_A, σ, ν) be a subshift of finite type whose stochastic matrix verifies that there exists n_0 such that all entries of M^{n_0} are positive. Let $\{t_n\}$ be a non decreasing sequence of natural numbers. Given a sequence $s_0 = (i_0, i_1, \ldots) \in \Sigma_A$, let $\widetilde{\mathcal{W}}(s_0, \{t_n\})$ be the set of sequences $s = (m_0, m_1, \ldots) \in \Sigma_A$ such that*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \ldots\ ,\ m_{n+t_n} = i_{t_n}\,, \qquad \textit{for infinitely many } n.$$

Then,

(1) *If $\sum_n p_{i_0} p_{i_0,i_1} \cdots p_{i_{t_n-1},i_{t_n}} = \infty$, then $\nu(\widetilde{\mathcal{W}}(s_0, \{t_n\})) = 1$. Besides, we have that*

$$\lim_{n\to\infty} \frac{\#\{j \le N :\ \sigma^j(s) \in C^{0,1,\ldots,t_j}_{i_0,i_1,\ldots,i_{t_j}}\}}{\sum_{n=1}^N p_{i_0} p_{i_0,i_1} \cdots p_{i_{t_n-1},i_{t_n}}} = 1\,, \qquad \textit{for } \nu\textit{-almost every } s \in \Sigma_A.$$

(2) *If $\sum_n p_{i_0} p_{i_0,i_1} \cdots p_{i_{t_n-1},i_{t_n}} < \infty$, then $\nu(\widetilde{\mathcal{W}}(s_0, \{t_n\})) = 0$.*

(3) *In any case we have that*

$$\frac{h}{h+\overline{L}} \le \mathrm{Dim}_{\mathbf{\Pi},\nu}(\widetilde{\mathcal{W}}(s_0, \{t_n\})) \le \min\left\{1, \frac{\log D}{h+\underline{L}}\right\}\,,$$

where $h = \sum_{i,j} p_i p_{i,j} \log(1/p_{i,j})$,

$$\underline{L} = \liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{p_{i_0} p_{i_0,i_1} \cdots p_{i_{t_n-1},i_{t_n}}} \qquad \textit{and} \qquad \overline{L} = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{p_{i_0} p_{i_0,i_1} \cdots p_{i_{t_n-1},i_{t_n}}}\,.$$

Proof. First, let us observe that for subshifts of finite type $X_0^+ = [0,1]$ because $\mathcal{P}_0$ is a finite partition of intervals. Also, since f preserves the Lebesgue measure λ we have that the ACIPM μ, whose existence is assured by Theorem E, coincides with λ. Then, parts (1) and (2) follow from Theorem 7.2. Finally any point in $[0,1]$ is an approximable point. Besides, from Lemma 4.4 we have that $\overline{\tau}_\lambda(x_0) = 0$ for $x_0 \in X_0^+ = [0,1]$. Therefore, part (3) follows from Remark 2.4, Theorem 6.2, Remark 6.5 and Proposition 6.1. □

The special properties of Bernoulli shifts allow us to get a better upper bound for the Hausdorff dimension of the set $\widetilde{\mathcal{W}}(s_0, \{t_n\})$ in that case. To prove it we will use the following concentration inequality (see, for example [26]).

Lemma (Hoeffding's tail inequality). *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $X_1, \ldots, X_n$ be independent copies of a bounded random variable X taking values in the interval (a, b) almost surely. Then, for any $t > 0$,*

$$\mu\Big[\sum_{i=1}^n X_i - n\,E(X) \ge t\Big] \le e^{-2t^2/(n(b-a)^2)}\,.$$

Theorem 7.3. *Let (Σ, σ, ν) be a Bernoulli shift, $\{t_n\}$ be a non decreasing sequence of natural numbers and $s_0 = (i_0, i_1, \ldots) \in \Sigma$. Then*

$$\frac{h}{h+\overline{L}} \le \mathrm{Dim}_{\mathbf{\Pi},\nu}(\widetilde{\mathcal{W}}(s_0, \{t_n\})) \le \frac{\sqrt{(h+\underline{L})^2 + 2\underline{L}(\log \frac{\max_j p_j}{\min_j p_j})^2} + h - \underline{L}}{\sqrt{(h+\underline{L})^2 + 2\underline{L}(\log \frac{\max_j p_j}{\min_j p_j})^2} + h + \underline{L}}\,.$$

where $h = \sum_i p_i \log(1/p_i)$,

$$\underline{L} = \liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{p_{i_0} p_{i_1} \cdots p_{i_{t_n}}} \qquad \textit{and} \qquad \overline{L} = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{p_{i_0} p_{i_1} \cdots p_{i_{t_n}}}\,.$$

In particular, if $p_0 = p_1 = \cdots = p_{D-1} = 1/D$ and $\overline{L} = \underline{L} = L$, then

$$\mathrm{Dim}_{\mathbf{\Pi},\nu}(\widetilde{\mathcal{W}}(s_0, \{t_n\})) = \frac{h}{h+L}\,.$$

Proof. The lower inequality follows from Corollary 7.1. So we only need to deal with the upper one. We define the collections

$$\mathcal{F}_n = \{C^{0,\ldots,n+t_n}_{j_0,\ldots,j_{n-1},i_0,\ldots,i_{t_n}} :\ j_0.j_1,\ldots,j_{n-1} \in \{0,\ldots,D-1\}\}\,.$$

Then, the collection

$$\mathcal{G}_N = \cup_{n=N}^{\infty} \mathcal{F}_n$$

covers the set $\widetilde{\mathcal{W}}(s_0,\{t_n\})$ and

$$\sum_{n=N}^{\infty} \sum_{F\in\mathcal{F}_n} \nu(F)^\tau = \sum_{n=N}^{\infty} \nu(C^{0,\ldots,t_n}_{i_0,\ldots,i_{t_n}})^\tau \sum_{j_0,\ldots,j_{n-1}=1}^{D-1} \nu(C^{0,\ldots,n-1}_{j_0,\ldots,j_{n-1}})^\tau\,. \tag{91}$$

For each n, we will divide the partition of Σ

$$\mathcal{P}_{n-1} = \{C^{0,\ldots,n-1}_{j_0,\ldots,j_{n-1}} :\ j_0.j_1,\ldots,j_{n-1} \in \{0,\ldots,D-1\}\}$$

in the following three subcollections:

$$\mathcal{P}_{n-1,\text{big}} = \left\{C\in\mathcal{P}_{n-1} :\ \nu(C) \geq e^{-nh}\right\}\,,$$

$$\mathcal{P}_{n-1,\text{middle}} = \left\{C\in\mathcal{P}_{n-1} :\ e^{-\beta n} < \frac{\nu(C)}{e^{-nh}} < 1\right\}\,,$$

$$\mathcal{P}_{n-1,\text{small}} = \left\{C\in\mathcal{P}_{n-1} :\ \frac{\nu(C)}{e^{-nh}} \leq e^{-\beta n}\right\}\,,$$

where $\beta = (1+\varepsilon)(1-\tau)/(\alpha K)$, $0<\alpha<1$ and $\varepsilon>0$. Then

$$\sum_{C\in\mathcal{P}_{n-1,\text{big}}} \nu(C)^\tau = \sum_{C\in\mathcal{P}_{n-1,\text{big}}} \frac{1}{\nu(C)^{1-\tau}}\,\nu(C) \leq e^{nh(1-\tau)} \tag{92}$$

and

$$\sum_{C\in\mathcal{P}_{n-1,\text{middle}}} \nu(C)^\tau \leq (\#\mathcal{P}_{n-1,\text{middle}})^{1-\tau} \left(\sum_{C\in\mathcal{P}_{n-1,\text{middle}}} \nu(C)\right)^\tau\,.$$

Since $(\#\mathcal{P}_{n-1,\text{middle}})\,e^{-(h+\beta)n} \leq \sum_{C\in\mathcal{P}_{n-1,\text{middle}}} \nu(C)$, we deduce that

$$\sum_{C\in\mathcal{P}_{n-1,\text{middle}}} \nu(C)^\tau \leq e^{n(h+\beta)(1-\tau)} \sum_{C\in\mathcal{P}_{n-1,\text{middle}}} \nu(C) \leq e^{n(h+\beta)(1-\tau)}\,. \tag{93}$$

For each $n\in\mathbf{N}$ we will choose an increasing sequence $\{a_{n,k}\}$, $a_{n,k}\to\infty$ as $k\to\infty$, with $a_{n,1}=\beta n$. Using this sequence we divide the collection $\mathcal{P}_{n-1,\text{small}}$ in the following way:

$$\mathcal{P}_{n-1,\text{small}} = \bigcup_{k=1}^{\infty} \mathcal{P}^k_{n-1,\text{small}}\,,\qquad \mathcal{P}^k_{n-1,\text{small}} = \{C\in\mathcal{P}_{n-1} :\ e^{-a_{n,k+1}} \leq \frac{\nu(C)}{e^{-nh}} \leq e^{-a_{n,k}}\}\,.$$

Then,

$$\sum_{C\in\mathcal{P}_{n-1,\text{small}}} \nu(C)^\tau = \sum_{k=1}^{\infty} \sum_{C\in\mathcal{P}^k_{n-1,\text{small}}} \nu(C)^\tau \leq \sum_{k=1}^{\infty} (\#\mathcal{P}^k_{n-1,\text{small}})^{1-\tau} \left(\sum_{C\in\mathcal{P}^k_{n-1,\text{small}}} \nu(C)\right)^\tau\,.$$

Since $(\#\mathcal{P}^k_{n-1,\text{small}})\,e^{-nh}e^{-a_{n,k+1}} \leq \sum_{C\in\mathcal{P}^k_{n-1,\text{small}}} \nu(C)$, we deduce that

$$\sum_{C\in\mathcal{P}_{n-1,\text{small}}} \nu(C)^\tau \leq e^{nh(1-\tau)} \sum_{k=1}^{\infty} e^{(1-\tau)a_{n,k+1}} \sum_{C\in\mathcal{P}^k_{n-1,\text{small}}} \nu(C)\,. \tag{94}$$

Now, observe that

$$\bigcup_{C\in\mathcal{P}^k_{n-1,\text{small}}} C = \left\{s\in\Sigma :\ a_{n,k} \leq \log\frac{1}{\nu(P(n-1,s))} - nh < a_{n,k+1}\right\}\,.$$

For each $j = 0, 1, \ldots$, let $Z_j : \Sigma \longrightarrow \mathbf{R}$ be the ramdom variable defined by

$$Z_j(i_0, i_1, \ldots) = \log \frac{1}{p_{i_j}}.$$

These ramdom variables are independent and identically distributed with expectated value

$$E(Z_j) = \sum_{i=0}^{D-1} p_i \log \frac{1}{p_i} = h.$$

Moreover,

$$S_n(s) := \sum_{j=0}^{n-1} Z_j(s) = \log \frac{1}{\nu(P(n-1, s))}$$

and therefore

$$\bigcup_{C \in \mathcal{P}^k_{n-1,\text{small}}} C = \{s \in \Sigma : \ a_{n,k} \le S_n(s) - E(S_n) < a_{n,k+1}\}.$$

Hence, from Hoeffding's tail inequality we have that, for all $\varepsilon > 0$,

$$\nu\Big(\bigcup_{C \in \mathcal{P}^k_{n-1,\text{small}}} C \Big) \le e^{-K a^2_{n,k}/n}, \qquad \text{with} \quad K = \frac{2}{\big(\log \frac{\max_j p_j}{\min_j p_j} + \varepsilon\big)^2}.$$

Using now (94) we get that

$$\sum_{C \in \mathcal{P}_{n-1,\text{small}}} \nu(C)^\tau \le e^{nh(1-\tau)} \sum_{k=1}^{\infty} e^{(1-\tau)a_{n,k+1}} e^{-K a^2_{n,k}/n}.$$

Notice that for $0 < \alpha < 1$ the sequence defined by

$$a_{n,k+1} = \frac{K\alpha}{1-\tau} \frac{a^2_{n,k}}{n}, \qquad a_{n,1} = \beta n,$$

verifies that

$$a_{n,k} = \frac{(1-\tau)n}{K\alpha} \Big(\frac{K\alpha}{1-\tau} \beta \Big)^{2^k} = \frac{(1-\tau)n}{K\alpha} (1+\varepsilon)^{2^k} \longrightarrow \infty, \qquad \text{as } k \to \infty$$

and therefore

$$\sum_{C \in \mathcal{P}_{n-1,\text{small}}} \nu(C)^\tau \le e^{nh(1-\tau)} \sum_{k=1}^{\infty} e^{-K(1-\alpha) a^2_{n,k}/n}.$$

But

$$\sum_{k=1}^{\infty} e^{-K(1-\alpha) a^2_{n,k}/n} \le \int_{\beta n}^{\infty} e^{-K(1-\alpha)x^2/n} dx \le \frac{\Gamma(1/2)}{2\sqrt{(1-\alpha)K}} \sqrt{n}.$$

Hence, for any $\eta > 0$ and n large enough we have

$$\sum_{C \in \mathcal{P}_{n-1,\text{small}}} \nu(C)^\tau \le e^{nh(1-\tau)(1+\eta)}. \tag{95}$$

Using now (91), (92), (93) and (95) we deduce, for N large enough,

$$\sum_{n=N}^{\infty} \sum_{F \in \mathcal{F}_n} \nu(F)^\tau \le 3 \sum_{n=N}^{\infty} \nu(C^{0,\ldots,t_n}_{i_0,\ldots,i_{t_n}})^\tau e^{n(h+\beta)(1-\tau)}.$$

Since, given $\varepsilon > 0$, for N large enough we have that $\nu(C^{0,\ldots,t_n}_{i_0,\ldots,i_{t_n}}) \le e^{-n(\underline{L}(s_0)-\varepsilon)}$ we conclude that

$$\sum_{n=N}^{\infty} \sum_{F \in \mathcal{F}_n} \nu(F)^\tau \le 3 \sum_{n=N}^{\infty} e^{-n\tau(\underline{L}(s_0)-\varepsilon)} e^{n(h+\beta)(1-\tau)} \to 0 \qquad \text{as } N \to \infty$$

if

$$\tau > \frac{\sqrt{(h+\underline{L}-\varepsilon)^2 + 4(1+\varepsilon)(\underline{L}-\varepsilon)/(K\alpha)} + h - \underline{L} + \varepsilon}{\sqrt{(h+\underline{L}-\varepsilon)^2 + 4(1+\varepsilon)(\underline{L}-\varepsilon)/(K\alpha)} + h + \underline{L} - \varepsilon}.$$

Therefore,

$$\mathrm{Dim}_{\mathbf{\Pi},\nu}(\widetilde{\mathcal{W}}(s_0,\{t_n\})) \leq \frac{\sqrt{(h+\underline{L}-\varepsilon)^2+2(1+\varepsilon)(\underline{L}-\varepsilon)(\log\frac{\max_j p_j}{\min_j p_j}+\varepsilon)^2/\alpha}+h-\underline{L}+\varepsilon}{\sqrt{(h+\underline{L}-\varepsilon)^2+2(1+\varepsilon)(\underline{L}-\varepsilon)(\log\frac{\max_j p_j}{\min_j p_j}+\varepsilon)^2/\alpha}+h+\underline{L}-\varepsilon}.$$

The result follows by taking $\varepsilon \to 0$ and $\alpha \to 1$.

□

The above results allow us to get, for example, the following one:

Corollary 7.2. *Let (Σ, σ, ν) be a Bernoulli shift.*

(1) *Let $t_n = [\log n]$. Then, for every sequence $(i_0, i_1, \dots) \in \Sigma$ we have that, for ν-almost all sequence $(m_0, m_1, \dots) \in \Sigma$,*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \dots\ ,\ m_{n+t_n} = i_{t_n}\,, \qquad \textit{for infinitely many } n.$$

(2) *Let $t_n = [n^\kappa]$ with $\kappa > 0$. Then, for every sequence $(i_0, i_1, \dots) \in \Sigma$, the set $\widetilde{\mathcal{W}}$ of sequences $(m_0, m_1, \dots) \in \Sigma$ such that*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \dots\ ,\ m_{n+t_n} = i_{t_n}\,, \qquad \textit{for infinitely many } n,$$

has zero ν-measure. Moreover, the ν-grid Hausdorff dimension of $\widetilde{\mathcal{W}}$ is 1 if $0 < \kappa < 1$ and zero if $\kappa > 1$.

Proof. Notice that if $t_n = [n^\kappa]$, we have

$$\sum_n p_{i_0}p_{i_1}\cdots p_{i_{t_n}} \leq \sum_n (\max_j p_j)^{t_n} \sum_n (\max_j p_j)^{n^\kappa - 1} < \infty$$

and if $t_n = [\log n]$ we have that

$$\sum_n p_{i_0}p_{i_1}\cdots p_{i_{t_n}} \geq \sum_n (\min_j p_j)^{t_n} \geq \sum_n (\min_j p_j)^{1+\log n} = \infty\,.$$

Also if $t_n = [n^\kappa]$ we have that $L = 0$ if $0 < \kappa < 1$ and $L = \infty$ if $\kappa > 1$. □

When $p_0 = \cdots = p_{D-1} = 1/D$ we can identify the Bernoulli shift (Σ, σ, ν) with the set of D-base representations of numbers in the interval $[0,1]$. The associated expanding map f is then the map $f(x) = Dx \pmod 1$, and the measure results contained in Corollary 7.2 in this particular case, are well known (see [35]).

7.1.2 Gauss transformation

Let us consider now the map $\phi : [0,1] \longrightarrow [0,1]$ given by

$$\phi(x) = \begin{cases} \dfrac{1}{x} - \left[\dfrac{1}{x}\right], & \text{if } x \neq 0\,, \\ 0\,, & \text{if } x = 0\,. \end{cases}$$

Here $[x]$ denotes the integer part of x. The map ϕ is called the *Gauss transformation* and it is very close related with the theory of continued fractions. Recall that given $0 < x < 1$ we can write it as

$$x = \frac{1}{n_0 + \phi(x)}\,, \qquad \text{with } n_0 := \left[\frac{1}{x}\right].$$

If $\phi(x) \neq 0$, i.e. if $x \notin \{1/n : n \in \mathbf{N}\} \cup \{0\}$, we can repeat the process with $\phi(x)$ to obtain

$$x = \cfrac{1}{n_0 + \cfrac{1}{n_1 + \phi^2(x)}}\,, \qquad \text{with } n_1 := \left[\frac{1}{\phi(x)}\right].$$

If $\phi^n(x) \neq 0$ for all n, or equivalently if x is irrational, we can repeat the process for all n and associate in this way to x the infinite sequence $\{n_j\}$, with $n_j = [1/\phi^j(x)]$ and we write

$$x := [n_0\ n_1\ n_2\ \dots] = \lim_{j\to\infty} \cfrac{1}{n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + n_j}}}}\,.$$

Observe that if we denote by I_n the interval $I_n = (1/(n+1), 1/n)$, then the sequence n_j is determined by the property $\phi^j(x) \in I_{n_j}$.

If x is rational the above expansion is finite (ending with n such that $\phi^n(x) = 0$. We call to the code $[n_0\ n_1\ n_2\ \dots]$ the continued fraction expansion of x. It is clear that the Gauss transformation acts on the continued fraction expansions as the left shift

$$x = [n_0\ n_1\ n_2\ \dots] \qquad \Longrightarrow \qquad \phi(x) = [n_1\ n_2\ \dots]\,.$$

It is not difficult to check that the Gauss transformation ϕ is a Markov transformation with respect to the partition $\mathcal{P}_0 = \{I_n\}$ and that the continued fraction expansion of x coincide with the code associated to an expanding map given in Section 4.1. It is also easy to check that ϕ preserves the so called Gauss measure which is defined by

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{1}{1+x}\, d\lambda(x)$$

where λ denotes Lebesgue measure. Since this measure is obviously absolutely continuous with respect to λ, we conclude that the Gauss measure is the unique ϕ-invariant absolutely continuous probability whose existence is assured by Theorem E.

The next theorem is an example of the kind of statements that we can obtain when we apply our results to the Gauss transformation.

Corollary 7.3.

(1) *If $\alpha > 1$ then, for almost all $x_0 \in [0,1]$, and more precisely, if $x_0 = [\,i_0\,,\ i_1\,,\ \dots]$ is an irrational number such that $\log i_n = o(n)$ as $n \to \infty$, we have that*

$$\liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = 0\,, \qquad \textit{for almost all } x \in [0,1].$$

(2) *If $\alpha < 1$, then for all $x_0 \in [0,1]$ we have that*

$$\liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = \infty\,, \qquad \textit{for almost all } x \in [0,1].$$

(3) *If x_0 verifies the same hypothesis than in part (1), then*

$$\mathrm{Dim}\left\{x \in [0,1] :\ \liminf_{n\to\infty} n^{1/\alpha}|\phi^n(x) - x_0| = 0\right\} = 1\,, \qquad \textit{for any } \alpha > 0.$$

and

$$\mathrm{Dim}\left\{x \in [0,1] :\ \liminf_{n\to\infty} e^{n\kappa}|\phi^n(x) - x_0| = 0\right\} \geq \frac{\pi^2}{\pi^2 + 6\kappa\log 2}\,, \qquad \textit{for any } \kappa > 0.$$

Proof. Let us observe first that now $\underline{\delta}_\lambda(x_0) = \overline{\delta}_\lambda(x_0) = 1$ for all $x_0 \in [0,1]$ and that obviously λ and μ are comparable in $[0,1]$. With this facts in mind, part (2) is a consequence of part (2) of Theorem 7.1 if $x_0 \in \cup_j I_j$. Part (2) is also true if $x_0 = 1/m$ for some $m \in \mathbf{N}$, since λ and μ are comparable in $[0,1]$ and then we do not need that $B(x_0, r_k) \subset P(0, x_0)$ in the proof of Proposition 5.1.

Since $T(P) = (0,1)$ for all $P \in \mathcal{P}_0$ we can use Proposition 4.2 for the case $j = n+1$ to get that

$$\frac{\lambda(P(n, x_0))}{\lambda(P(n+1, x_0))} \asymp \frac{1}{\lambda(P(0, T^{n+1}(x_0)))}\,.$$

But $T^{n+1}(x_0) = [\, i_{n+1}\,,\ i_{n+2}\,,\ \dots]$ and therefore $P(0, T^{n+1}(x_0)) = (1/(i_{n+1}+1), 1/i_{n+1})$. Hence

$$\log \frac{\lambda(P(n,x_0))}{\lambda(P(n+1,x_0))} \asymp \log i_{n+1}$$

and we conclude that $\tau(x_0) = 0$ if $\log i_n = o(n)$ as $n \to \infty$. Part (1) follows now from Corollary 5.4, since in this case the set X_0 is precisely the set of irrational numbers in $[0,1]$.

Since λ and μ are comparable in $[0,1]$ we have that all irrational number is approximable (see Definition 6.1) and as we have just seen $\tau(x_0) = 0$ if $\log i_n = o(n)$, we can use Remark 2.4 and Corollary 6.2 to obtain that

$$\mathrm{Dim}\left\{ x \in [0,1] : \ \liminf_{n\to\infty} \frac{|\phi^n(x) - x_0|}{r_n} = 0 \right\} \geq \frac{h}{h+\ell}$$

for any non increasing sequence $\{r_n\}$ of positive numbers such that there exists $\ell := \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{r_n}$. Here h denotes de entropy of the Gauss transformation which is known to be

$$h = \frac{2}{\log 2} \int_0^1 \frac{\log(1/x)}{1+x}\, dx = \frac{\pi^2}{6 \log 2}\,.$$

Part (3) follows now from the fact that if $r_n = n^{-1/\alpha}$ with $\alpha > 0$ then $\ell = 0$, and if $r_n = e^{-n\kappa}$ with $\kappa > 0$ we have $\ell = \kappa$. $\square$

For continued fractions expansions there is an analogous to Corollary 7.2. However we have preferred to state the following result involving the digits appearing in the continued fraction expansion of x_0.

Corollary 7.4. *Let $x_0 \in [0,1]$ be an irrational number with continued fraction expansion $x_0 = [\, i_0, i_1, \dots]$ and let t_n be a non decreasing sequence of natural numbers. Let $\widetilde{W}$ be the set of points $x = [\, m_0, m_1, \dots] \in [0,1]$ such that*

$$m_n = i_0\,,\ m_{n+1} = i_1\,,\ \dots\ ,\ m_{n+t_n} = i_{t_n}\,, \qquad \textit{for infinitely many } n.$$

(1) $\lambda(\widetilde{W}) = 1$, *if*

$$\sum_n \frac{1}{(i_0+1)^2 \cdots (i_{t_n}+1)^2} = \infty\,.$$

(2) $\lambda(\widetilde{W}) = 0$, *if*

$$\sum_n \frac{1}{i_0^2 \cdots i_{t_n}^2} < \infty\,.$$

(3) *In any case, if* $\log i_n = o(n)$ *as* $n \to \infty$*, then*

$$\mathrm{Dim}(\widetilde{W}) \geq \frac{h}{h + \limsup_{n\to\infty} \frac{1}{n} \log(i_0+1)^2 \cdots (i_{t_n}+1)^2}\,.$$

Proof. It is easy to check that, for all $n \in \mathbf{N}$,

$$\frac{1}{(i_0+1)^2 \cdots (i_n+1)^2} \leq \lambda(P(n,x_0)) \leq \frac{1}{i_0^2 \cdots i_n^2}\,.$$

Then, parts (1) and (2) follow from Theorem 7.2 and part (3) is a consequence of Theorem 6.2 and Remark 2.4. $\square$

7.2 Inner functions

A Blaschke product is a complex function of the type

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{z - a_k}{1 - \overline{a_k} z}\,, \qquad |a_k| < 1\,,$$

verifying the Blaschke condition $\sum_{k=1}^{\infty} (1 - |a_k|) < \infty$. The function $B(z)$ is holomorphic in the unit disk $\mathbf{D} = \{z : |z| < 1\}$ of the complex plane and it is an example of an inner function, i.e a holomorphic function f defined on $\mathbf{D}$ and with values in $\mathbf{D}$ whose radial limits

$$f^*(\xi) := \lim_{r\to 1^-} f(r\xi)$$

(which exists for almost every ξ by Fatou's Theorem) have modulus 1 for almost every $\xi \in \partial\mathbf{D}$. Therefore an inner function $f(z)$ induces a mapping $f^* : \partial\mathbf{D} \longrightarrow \partial\mathbf{D}$. It is well known that any inner function can be written as

$$f(z) = e^{i\phi} B(z) \exp\left(- \int_{\partial\mathbf{D}} \frac{\xi + z}{\xi - z} \, d\nu(\xi) \right)$$

where $B(z)$ is a Blaschke product and ν is a finite positive singular measure on $\partial\mathbf{D}$.

For inner functions it is well known the following result, see e.g. [39]:

Theorem G (Löwner's lemma). *If $f : \mathbf{D} \longrightarrow \mathbf{D}$ is an inner function then $f^* : \partial\mathbf{D} \longrightarrow \partial\mathbf{D}$ preserves Lebesgue measure if and only if $f(0) = 0$.*

We recall that, by the Denjoy-Wolff theorem [14], for any holomorphic function $f : \mathbf{D} \longrightarrow \mathbf{D}$ which is not conjugated to a rotation, there exists a point $p \in \overline{\mathbf{D}}$, the so called Denjoy-Wolff point of f, such the iterates f^n converge to p uniformly on compact subsets of $\mathbf{D}$. Also, if $p \in \mathbf{D}$ then $f(p) = p$ and if $p \in \partial\mathbf{D}$ then $f^*(p) = p$. Hence, if f is an inner function which is not conjugated to a rotation and does not have a fixed point $p \in \mathbf{D}$ then its Denjoy-Wolff point p belongs to $\partial\mathbf{D}$ and f^n converges to p uniformly on compact subsets of $\mathbf{D}$. Bourdon, Matache and Shapiro [9] and Poggi-Corradini [38] have proved independently that if f is inner with a fixed point in $p \in \partial\mathbf{D}$, then $(f^*)^n$ can converge to p for almost every point in $\partial\mathbf{D}$. In fact, see Theorem 4.2 in [9], $(f^*)^n \to p$ almost everywhere in $\partial\mathbf{D}$ if and only if $\sum_n (1 - |f^n(0)|) < \infty$.

If f is inner with a fixed point in $\mathbf{D}$, f preserves the harmonic measure ω_p. We recall that ω_p can be defined as the unique probability measure such that, for all continuous function $\phi : \partial\mathbf{D} \longrightarrow \mathbf{R}$,

$$\int_{\partial\mathbf{D}} \phi \, d\omega_p = \widetilde{\phi}(p) \,,$$

where $\widetilde{\phi}$ is the unique extension of ϕ which is continuous in $\overline{\mathbf{D}}$ and harmonic in $\mathbf{D}$. It follows that if A is an arc in $\partial\mathbf{D}$, then $\omega_p(A)$ is the value at the point p of the harmonic function whose radial limits take the value 1 on A and the value 0 on the exterior of A.

If f is inner with a fixed point in $\mathbf{D}$, but it is not conjugated to a rotation, J. Aaronson [1] and J.H. Neuwirth [34] proved, independently, that f^* is exact with respect to harmonic measure and therefore mixing and ergodic. In fact, inner functions are also ergodic with respect to α-capacity [21]. An interesting study of some dynamical properties of inner functions is contained in the works of M. Craizer. In [11] he proves that if f' belongs to the Nevanlinna class, then the entropy of f^* is finite and it can be calculated by the formula

$$h(f^*) = \frac{1}{2\pi} \int_0^{2\pi} \log |(f^*)'(x)| \, dx \,,$$

where $(f^*)'$ denotes the angular derivative of f. He also proves that the Rohlin invertible extension of an inner function with a fixed point in $\mathbf{D}$ is equivalent to a generalized Bernoulli shift, see [12].

The mixing properties of inner functions are even stronger. In this sense Ch. Pommerenke [36] has shown the following

Theorem H (Ch. Pommerenke). *Let $f : \mathbf{D} \longrightarrow \mathbf{D}$ be an inner function with $f(0) = 0$, but not a rotation. Then, there exists a positive absolute constant K such that*

$$\left| \frac{\lambda[B \cap (f^*)^{-n}(A)]}{\lambda(A)} - \lambda(B) \right| \leq K \, e^{-\alpha n} \,,$$

for all $n \in \mathbf{N}$, for all arcs $A, B \subset \partial\mathbf{D}$, where $\alpha = \max\{1/2, |f'(0)|\}$ and λ denotes normalized Lebesgue measure.

In the terminology of [19] this imply that inner functions with $f(p) = p$ ($p \in \mathbf{D}$) are uniformly mixing at any point of $\partial\mathbf{D}$ with respect to the harmonic measure ω_p. In particular, we have that the correlation coefficients of characteristic functions *of balls* have exponential decay. As a consequence of Theorem 3 in [19], and the arguments of the proofs of Corollaries 5.2 and 5.4 we have that if ξ_0 is any point in $\partial\mathbf{D}$ and $\{r_n\}$ is a non increasing sequence of positive numbers, then we have that

(A) If $\sum_{n=1}^{\infty} r_n < \infty$, then

$$\liminf_{n\to\infty} \frac{d((f^*)^n(\xi),\xi_0)}{r_n} = \infty\,, \qquad \text{for almost every } \xi \in \partial\mathbf{D}\,.$$

(B) If $\sum_{n=1}^{\infty} r_n = \infty$, then

$$\lim_{N\to\infty} \frac{\#\{n \le N:\ d((f^*)^n(\xi),\xi_0) < r_n\}}{\sum_{n=1}^{N} r_n} = 1\,, \qquad \text{for almost every } \xi \in \partial\mathbf{D}\,.$$

and

$$\liminf_{n\to\infty} \frac{d((f^*)^n(\xi),\xi_0)}{r_n} = 0\,, \qquad \text{for almost every } \xi \in \partial\mathbf{D}\,.$$

A finite Blaschke product B (with, say, N factors) is a rational function of degree N and therefore it is a covering of order N of $\partial\mathbf{D}$. As a consequence B has a fixed point in $\partial\mathbf{D}$ if $N \ge 3$ or if $N = 2$ and $B(0) = 0$. Hence, we can choose a branch of the argument of $B(e^{i\theta})$ mapping 0 on 0 and $[0, 2\pi]$ onto $[0, 2N\pi]$. Also $B(z)$ is C^∞ at the boundary $\partial\mathbf{D}$ of the unit disk and its derivative verifies

$$|B'(z)| = \sum_{k=1}^{N} \frac{1-|a_k|^2}{|z-a_k|^2}\,, \qquad \text{if } |z| = 1\,.$$

Therefore, if $B(0) = 0$, we have that $|B'(z)| > C > 1$ for all $z \in \partial\mathbf{D}$, and the dynamic of B^* on $\partial\mathbf{D}$ is isomorphic to the dynamic of a Markov transformation with a finite partition $\mathcal{P}_0$ (it has N elements) and having the Bernoulli property. Besides, since the Lebesgue measure is exact we have that the ACIPM measure of the system is precisely Lebesgue measure λ. Hence, we obtain the following improvement of statement (A):

Theorem 7.4. *Let $B : \mathbf{D} \longrightarrow \mathbf{D}$ be a finite Blaschke product with a fixed point $p \in \mathbf{D}$, but not an automorphism which is conjugated to a rotation. Let also ξ_0 be any point in $\partial\mathbf{D}$ and let $\{r_n\}$ be a non increasing sequence of positive numbers. Then*

$$\operatorname{Dim}\left\{\xi \in \partial\mathbf{D} : \liminf_{n\to\infty} \frac{d((B^*)^n(\xi),\xi_0)}{r_n} = 0\right\} \ge \frac{h}{h+\overline{\ell}}$$

where $\overline{\ell} = \limsup_{n\to\infty} \frac{1}{n}\log\frac{1}{r_n}$, $h = \int_{\partial\mathbf{D}} \log|B'(z)|\,d\lambda(z)$ and Dim *denotes Hausdorff dimension. The result is sharp in the sense that we get equality when $B(z) = z^N$ and $\overline{\ell} = \ell = \lim_{n\to\infty} \frac{1}{n}\log\frac{1}{r_n}$.*

Proof. In the case that $p = 0$ the result follows from the above comments and Theorem 7.1. In the general case, let $T : D \longrightarrow \mathbf{D}$ be a Möbius transformation such that $T(p) = 0$. Then, $g = T \circ B \circ T^{-1}$ is a finite Blasckhe product with $g(0) = 0$. Besides, it is easy to see that

$$\{\xi \in \partial\mathbf{D}:\ d((g^*)^n(\xi),\xi_0) < r_n \text{ i.o.}\} \subseteq T(\{\xi \in \partial\mathbf{D}:\ d((B^*)^n(\xi),T^{-1}(\xi_0)) < Cr_n \text{ i.o.}\})$$

where C is a constant depending on T. Therefore the lower bound follows from the case $p = 0$. The equality for $B(z) = z^N$ follows from Proposition 6.2. ☐

Theorem 7.4 is also true for the following infinite Blaschke product:

$$B(z) = \prod_{k=0}^{\infty} \frac{z-a_k}{1-a_k z}\,, \qquad a_k = 1 - 2^{-k}\,.$$

since as we will see, the dynamic of B^* on $\partial\mathbf{D}$ is isomorphic to the dynamic of a Markov transformation with a countable partition $\mathcal{P}_0$ and with the Bernoulli property. Notice also that B^* is exact with respect to Lebesgue measure and therefore we have that the ACIPM measure is Lebesgue measure.

For this Blaschke product B is defined in $\partial\mathbf{D} \setminus \{1\}$ and in fact it is C^∞ there and

$$|B'(z)| = \sum_{k=0}^{\infty} \frac{1-a_k^2}{|z-a_k|^2}\,, \qquad \text{if } |z| = 1,\ z \ne 1\,.$$

If we denote $B(e^{2\pi it}) = e^{2\pi iS(t)}$ then $S'(t) = |B'(e^{2\pi it})| > C > 1$. Moreover, it follows from Phragmén-Lindelöf Theorem that the image of $S(t)$ is $(-\infty, \infty)$ and so we can define the intervals

$P_j = \{t \in (0,1) : \; j < S(t) < j+1\}$. The transformation $T : [0,1] \longrightarrow [0,1]$ given by $T(t) = S(t) \pmod 1$, $T(0) = T(1) = 0$, is a Markov transformation with partition $\mathcal{P}_0 = \{P_j\}$. To see this we only left to prove property (f). We define the following collection of subarcs of $\partial\mathbf{D}$: $I_k^+ = \{e^{i\alpha} : \theta_{k+1} < \alpha < \theta_k\}$ $(k \geq 0)$, where $\theta_0 = \pi$ and for each $k \geq 1$ we denote by $e^{i\theta_k}$ $(\theta_k \in (0,\pi))$ the point whose distance to 1 is $1 - a_{k-1} = 2^{-(k-1)}$. We define also $I_k^- = \{z \in \partial\mathbf{D} : \bar{z} \in I_k^+\}$. It is geometrically clear that if $z \in I_j^{\pm}$, then $|\sin 2\pi t| \leq C\,2^{-j}$ and also that

$$|z - a_k| \geq \begin{cases} C\,2^{-j}\,, & \text{for } k \geq j \\ C\,2^{-k}\,, & \text{for } k < j\,, \end{cases}$$

Now, if $z = e^{2\pi i t} \in I_j^{\pm}$, we have that

$$S'(t) = |B'(e^{2\pi i t})| = \sum_{k=0}^{\infty} \frac{1 - a_k^2}{|e^{2\pi i t} - a_k|^2} \geq C\,\frac{1 - a_j}{2^{-2j}} = C\,2^j\,,$$

$$S'(t) = \sum_{k=0}^{\infty} \frac{1 - a_k^2}{|e^{2\pi i t} - a_k|^2} \leq C \sum_{k<j} \frac{2^{-k}}{2^{-2k}} + C \sum_{k\geq j} \frac{2^{-k}}{2^{-2j}} \leq C\,2^j$$

and

$$|S''(t)| \leq C \sum_{k=0}^{\infty} \left| \frac{a_k(1 - a_k^2)\sin 2\pi t}{(e^{2\pi i t} - a_k)^4} \right| \leq C \sum_{k<j} \frac{2^{-k}2^{-j}}{2^{-4k}} + C \sum_{k\geq j} \frac{2^{-k}2^{-j}}{2^{-4j}} \leq C\,2^{2j}\,.$$

Therefore, since $\lambda(I_j^{\pm}) \asymp 2^{-j}$ we have that $\int_{I_j^{\pm}} S'(t)\,dt \asymp C$ and so each $P_j \in \mathcal{P}_0$ contains at most a fixed constant number of consecutive intervals $I_k^{\pm}$. Hence, there exists an absolute constant C such that, if $t_1, t_2, t_3 \in P_j$, then

$$\frac{|T''(t_1)|}{T'(t_2)\,T'(t_3)} \leq C$$

and this implies that T verify property (f) of Markov transformations.

Finally, the entropy h of B^* (or $T(t)$) is finite, because

$$\begin{aligned} h = \int_{\partial\mathbf{D}} \log|B'(z)|\,d\lambda(z) &= 2\sum_{j=0}^{\infty} \int_{I_j} \log|B'(z)|\,d\lambda(z) \\ &\leq 2\sum_{j=0}^{\infty} \log\left(C\sum_{k=0}^{\infty} \frac{2^{-k}}{2^{-2j}}\right)\frac{1}{2^{j+1}} = 2\sum_{j=0}^{\infty} \log\left(C2^{2^j}\right)\frac{1}{2^{j+1}} < \infty\,. \end{aligned}$$

The singular inner functions

$$f(z) = e^{c\frac{1+z}{1-z}}\,, \qquad \text{for } c < -2.$$

also verify Theorem 7.4. These inner functions have only one singularity at $z = 1$ and its Denjoy-Wolff point p is real and it verifies $0 < p < 1$. It is easy to see that if $f(e^{2\pi i t}) = e^{2\pi i S(t)}$ for $t \in [0,1]$, then $S(t) = \frac{c}{2\pi}\cot \pi t$. and the dynamic of f^* on $\partial\mathbf{D}$ is isomorphic to the dynamic of the Markov transformation $T(t) = S(t) \pmod 1$. We have that the partition $\mathcal{P}_0$ for T is countable, $\mathcal{P}_0 = \{P_j : j \in \mathbf{Z}\}$ where $P_j = \{t \in (0,1) : \; j < S(t) < j+1\}$, and T has the Bernoulli property, i.e. $T(P_j) = (0,1)$. Notice also that $T'(t) = \frac{|c|}{2}\csc^2 \pi t > 1$ and that, for $x, y \in P_j$,

$$\left|\frac{T'(x)}{T'(y)} - 1\right| = \frac{|T(x) + T(y)|}{T(y)^2 + (c/2\pi)^2}\,|T(x) - T(y)| \leq \frac{2j+2}{j^2 + (c/2\pi)^2}\,|T(x) - T(y)| \leq C\,|T(x) - T(y)|\,.$$

It is known that the entropy of f is finite (see [31])

$$h(f) = \log\left(\frac{1}{1-p^2}\log\frac{1}{p^2}\right) < \infty\,.$$

More generally, Theorem 7.4 holds for inner functions f with a fixed point $p \in \mathbf{D}$ and finite entropy such that the transformation T defined as in these examples is Markov. This happens, for example, if the set of singularities of f in $\partial\mathbf{D}$ is finite, the lateral limits of f^* at the singular points are $\pm\infty$ and T verifies properties (d) and (f) of Markov transformations. Notice that the condition on the lateral limits holds, for example, for Blasckhe products whose singular set is finite and each singular point ξ is an accumulation point of zeroes inside of a Stolz cone with vertex ξ. However we think that Theorem 7.4 is true for any inner function with a fixed point $p \in \mathbf{D}$ and finite entropy.

7.3 Expanding endomorphisms

Let M be a compact Riemannian manifold. A C^1 map $f : M \longrightarrow M$ is an *expanding endomorphism* if there exists a natural number $n \geq 1$ and constants $C > 0$ and $\beta > 1$ such that

$$\|(D_x f^n)u\| > C\,\beta^n \|u\|\,, \qquad \text{for all } x \in M, u \in T_x M\,.$$

A C^1 expanding endomorphism of a compact connected Riemannian manifold M whose derivative $D_x f$ is a Hölder continuous function of x is an expanding map with respect to Lebesgue measure λ and a finite Markov partition $\mathcal{P}_0$, see [30], p.171. Therefore, the unique f-invariant probability measure whose existence is assured by Theorem E is comparable to λ in the whole M. Our results also apply for this dynamical system.

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